Problems in Introductory Physics

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Contents

1 Measurement

This is not a textbook. It's a book of problems meant to be used along with a textbook. Although each chapter of this book starts with a brief summary of the relevant physics, that summary is not meant to be enough to allow the reader to actually learn the subject from scratch. The purpose of the summary is to show what material is needed in order to do the problems, and to show what terminology and notation are being used.

1.1 The SI

The Système International (SI) is a system of measurement in which mechanical quantities are expressed in terms of three basic units: the meter (m), the kilogram (kg), and the second (s). Other units can be built out of these. For example, the SI unit to measure the flow of water through a pipe would be kg/s.

To modify the units there is a consistent set of prefixes. The following are common and should be memorized:

prefix		meaning
nano-	n	10^{-9}
micro-	μ	10^{-6}
milli-	m	10^{-3}
kilo-	k	10^{3}
mega-	M	10^{6}

The symbol μ, for micro-, is Greek lowercase mu, which is equivalent to the Latin "m." There is also centi-, 10^{-2}, which is only used in the centimeter.

1.2 Significant figures

The international governing body for football ("soccer" in the US) says the ball should have a circumference of 68 to 70 cm. Taking the middle of this range and dividing by π gives a diameter of approximately 21.9633821466815563361059593454069819 6 cm. The digits after the first few are completely meaningless. Since the circumference could have varied by about a centimeter in either direction, the diameter is fuzzy by something like a third of a centimeter. We say that the additional, random digits are not *significant figures*. If you write down a number with a lot of gratuitous insignificant figures, it shows a lack of scientific literacy and imples to other people a greater precision than you really have.

As a rule of thumb, the result of a calculation has as many significant figures, or "sig figs," as the least accurate piece of data that went in. In the example with the soccer ball, it didn't do us any good to know π to dozens of digits, because the bottleneck in the precision of the result was the figure for the circumference, which was two sig figs. The result is 22 cm. The rule of thumb works best for multiplication and division.

The numbers 13 and 13.0 mean different things, because the latter implies higher precision. The number 0.0037 is two significant figures, not four, because the zeroes after the decimal place are placeholders. A number like 530 could be either two sig figs or three; if we wanted to remove the ambiguity, we could write it in scientific notation as 5.3×10^2 or 5.30×10^2.

1.3 Proportionalities

Often it is more convenient to reason about the ratios of quantities rather than their actual values. For example, suppose we want to know what happens to the area of a circle when we triple its radius. We know that $A = \pi r^2$, but the factor of π is not of interest here because it's present in both cases, the small circle and the large one. Throwing away the constant of proportionality, we can write $A \propto r^2$, where the proportionality symbol $\propto$, read "is proportional to," says that the left-hand side doesn't necessarily equal the right-hand side, but it does equal the right-hand side multiplied by a constant.

Any proportionality can be interpreted as a statement about ratios. For example, the statement $A \propto r^2$ is exactly equivalent to the statement that $A_1/A_2 = (r_1/r_2)^2$, where the subscripts 1 and 2 refer to any two circles. This in our example, the given information that $r_1/r_2 = 3$ tells us that $A_1/A_2 = 9$.

In geometrical applications, areas are always proportional to the square of the linear dimensions, while volumes go like the cube.

1.4 Estimation

It is useful to be able to make rough estimates, e.g., how many bags of gravel will I need to fill my driveway? Sometimes all we need is an estimate so rough that we only care about getting the result to about the nearest factor of ten, i.e., to within an order of magnitude. For example, anyone with a basic knowledge of US geography can tell that the distance from New Haven to New York is probably something like 100 km, not 10 km or 1000 km. When making estimates of physical quantities, the following guidelines are helpful:

1. Don't even attempt more than one significant figure of precision.

2. Don't guess area, volume, or mass directly. Guess linear dimensions and get area, volume, or mass from them. Mass is often best found by estimating linear dimensions and density.

3. When dealing with areas or volumes of objects with complex shapes, idealize them as if they were some simpler shape, a cube or a sphere, for example.

4. Check your final answer to see if it is reasonable. If you estimate that a herd of ten thousand cattle would yield 0.01 m^2 of leather, then you have probably made a mistake with conversion factors somewhere.

Problems

1-a1 Convert 134 mg to units of kg, writing your answer in scientific notation.

▷ Solution, p. 199

1-a2 Express each of the following quantities in micrograms:
(a) 10 mg, (b) 10^4 g, (c) 10 kg, (d) 100×10^3 g, (e) 1000 ng. ✓

1-a3 In the last century, the average age of the onset of puberty for girls has decreased by several years. Urban folklore has it that this is because of hormones fed to beef cattle, but it is more likely to be because modern girls have more body fat on the average and possibly because of estrogen-mimicking chemicals in the environment from the breakdown of pesticides. A hamburger from a hormone-implanted steer has about 0.2 ng of estrogen (about double the amount of natural beef). A serving of peas contains about 300 ng of estrogen. An adult woman produces about 0.5 mg of estrogen per day (note the different unit!). (a) How many hamburgers would a girl have to eat in one day to consume as much estrogen as an adult woman's daily production? (b) How many servings of peas? ✓

1-d1 The usual definition of the mean (average) of two numbers a and b is $(a+b)/2$. This is called the arithmetic mean. The geometric mean, however, is defined as $(ab)^{1/2}$ (i.e., the square root of ab). For the sake of definiteness, let's say both numbers have units of mass. (a) Compute the arithmetic mean of two numbers that have units of grams. Then convert the numbers to units of kilograms and recompute their mean. Is the answer consistent? (b) Do the same for the geometric mean. (c) If a and b both have units of grams, what should we call the units of ab? Does your answer make sense when you take the square root? (d) Suppose someone proposes to you a third kind of mean, called the superduper mean, defined as $(ab)^{1/3}$. Is this reasonable?

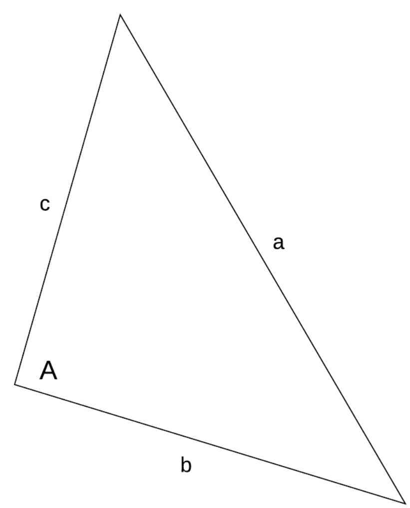
▷ Solution, p. 199

1-d2 (a) Based on the definitions of the sine, cosine, and tangent, what units must they have? (b) A cute formula from trigonometry lets you find any angle of a triangle if you know the lengths of its sides. Using the notation shown in the figure, and letting $s=(a+b+c)/2$ be half the perimeter, we have

$$\tan A/2 = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$$

Show that the units of this equation make sense. In other words, check that the units of the right-hand side are the same as your answer to part a of the question.

▷ Solution, p. 199



Problem 1-d2.

1-d3 Jae starts from the formula $V=\frac{1}{3}Ah$ for the volume of a cone, where A is the area of its base, and h is its height. He wants to find an equation that will tell him how tall a conical

tent has to be in order to have a certain volume, given its radius. His algebra goes like this:

$$
\begin{aligned}
V &= \frac{1}{3}Ah \\
A &= \pi r^2 \\
V &= \frac{1}{3}\pi r^2 h \\
h &= \frac{\pi r^2}{3V}
\end{aligned}
$$

Use units to check whether the final result makes sense. If it doesn't, use units to locate the line of algebra where the mistake happened.

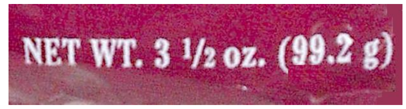
▷ Solution, p. 199

1-d4 The distance to the horizon is given by the expression $\sqrt{2rh}$, where r is the radius of the Earth, and h is the observer's height above the Earth's surface. (This can be proved using the Pythagorean theorem.) Show that the units of this expression make sense. Don't try to prove the result, just check its units. (For an example of how to do this, see problem 1-d3 on p. 9, which has a solution given in the back of the book.)

1-d5 Let the function x be defined by $x(t) = Ae^{bt}$, where t has units of seconds and x has units of meters. (For $b < 0$, this could be a fairly accurate model of the motion of a bullet shot into a tank of oil.) Show that the Taylor series of this function makes sense if and only if A and b have certain units.

1-g1 In an article on the SARS epidemic, the May 7, 2003 New York Times discusses conflicting estimates of the disease's incubation period (the average time that elapses from infection to the first symptoms). "The study estimated it to be 6.4 days. But other statistical calculations ... showed that the incubation period could be as long as 14.22 days." What's wrong here?

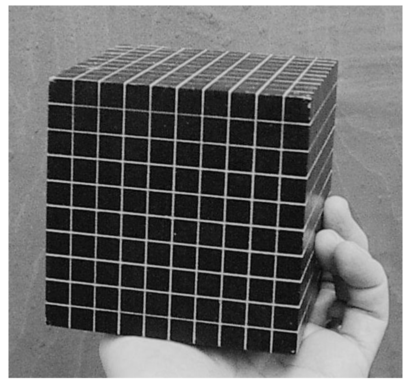
1-g2 The photo shows the corner of a bag of pretzels. What's wrong here?



Problem 1-g2.

1-j1 The one-liter cube in the photo has been marked off into smaller cubes, with linear dimensions one tenth those of the big one. What is the volume of each of the small cubes?

▷ Solution, p. 199



Problem 1-j1.

1-j2 How many cm^2 is 1 mm^2?

▷ Solution, p. 199

1-j3 Compare the light-gathering powers of a 3-cm-diameter telescope and a 30-cm telescope.

▷ Solution, p. 199

1-j4 The traditional Martini glass is shaped like a cone with the point at the bottom. Suppose you make a Martini by pouring vermouth into the glass to a depth of 3 cm, and then adding gin to bring the depth to 6 cm. What are the proportions of gin and vermouth?

▷ Solution, p. 199

1-j5 How many cubic inches are there in a cubic foot? The answer is not 12.

$\checkmark$

1-j6 Assume a dog's brain is twice as great in diameter as a cat's, but each animal's brain cells are the same size and their brains are the same shape. In addition to being a far better

companion and much nicer to come home to, how many times more brain cells does a dog have than a cat? The answer is not 2.

1-k1 One step on the Richter scale corresponds to a factor of 100 in terms of the energy absorbed by something on the surface of the Earth, e.g., a house. For instance, a 9.3-magnitude quake would release 100 times more energy than an 8.3. The energy spreads out from the epicenter as a wave, and for the sake of this problem we'll assume we're dealing with seismic waves that spread out in three dimensions, so that we can visualize them as hemispheres spreading out under the surface of the earth. If a certain 7.6-magnitude earthquake and a certain 5.6-magnitude earthquake produce the same amount of vibration where I live, compare the distances from my house to the two epicenters.

▷ Solution, p. 199

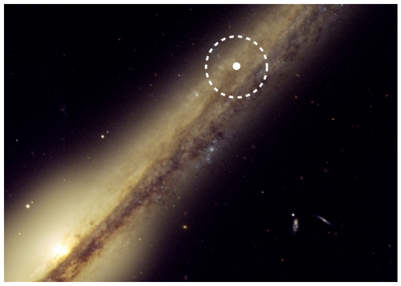
1-k2 The central portion of a CD is taken up by the hole and some surrounding clear plastic, and this area is unavailable for storing data. The radius of the central circle is about 35% of the outer radius of the data-storing area. What percentage of the CD's area is therefore lost? $\surd$

1-k3 The Earth's surface is about 70% water. Mars's diameter is about half the Earth's, but it has no surface water. Compare the land areas of the two planets. $\surd$

1-k4 At the grocery store you will see oranges packed neatly in stacks. Suppose we want to pack spheres as densely as possible, so that the greatest possible fraction of the space is filled by the spheres themselves, not by empty space. Let's call this fraction f. Mathematicians have proved that the best possible result is $f \approx 0.7405$, which requires a systematic pattern of stacking. If you buy ball bearings or golf balls, however, the seller is probably not going to go to the trouble of stacking them neatly. Instead they will probably pour the balls into a box and vibrate the box vigorously for a while to make them settle. This results in a random packing. The closest random packing has $f \approx 0.64$. Suppose that golf balls, with a standard diameter of 4.27 cm, are sold in bulk with the closest random packing. What is the diameter of the largest ball that could be sold in boxes of the same size, packed systematically, so that there would be the same number of balls per box? $\surd$

Problem 1-k4.

1-k5 Radio was first commercialized around 1920, and ever since then, radio signals from our planet have been spreading out across our galaxy. It is possible that alien civilizations could detect these signals and learn that there is life on earth. In the 90 years that the signals have been spreading at the speed of light, they have created a sphere with a radius of 90 light-years. To show an idea of the size of this sphere, I've indicated it in the figure as a tiny white circle on an image of a spiral galaxy seen edge on. (We don't have similar photos of our own Milky Way galaxy, because we can't see it from the outside.) So far we haven't received answering signals from aliens within this sphere, but as time goes on, the sphere will expand as suggested by the dashed outline, reaching more and more stars that might harbor extraterrestrial life. Approximately what year will it be when the sphere has expanded to fill a volume 100 times greater than the volume it fills today in 2010? $\surd$



Problem 1-k5.

1-k6 X-ray images aren't only used with human subjects but also, for example, on insects and flowers. In 2003, a team of researchers at Argonne National Laboratory used x-ray imagery to find for the first time that insects, although they do not have lungs, do not necessarily breathe completely passively, as had been believed previously; many insects rapidly compress and expand their trachea, head, and thorax in order to force air in and out of their bodies. One difference between x-raying a human and an insect is that if a medical x-ray machine was used on an insect, virtually 100% of the x-rays would pass through its body, and there would be no contrast in the image produced. Less penetrating x-rays of lower energies have to be used. For comparison, a typical human body mass is about 70 kg, whereas a typical ant is about 10 mg. Estimate the ratio of the thicknesses of tissue that must be penetrated by x-rays in one case compared to the other.

$\checkmark$

1-m1 A taxon (plural taxa) is a group of living things. For example, *Homo sapiens* and *Homo neanderthalensis* are both taxa — specifically, they are two different species within the genus *Homo*. Surveys by botanists show that the number of plant taxa native to a given contiguous land area A is usually approximately proportional to $A^{1/3}$. (a) There are 70 different species of lupine native to Southern California, which has an area of about $200,000\ \text{km}^2$. The San Gabriel Mountains cover about $1,600\ \text{km}^2$. Suppose that you wanted to learn to identify all the species of lupine in the San Gabriels. Approximately how many species would you have to familiarize yourself with? $\checkmark$
(b) What is the interpretation of the fact that the exponent, 1/3, is less than one?

$\star$

1-m2 The population density of Los Angeles is about 4000 people/km^2. That of San Francisco is about 6000 people/km^2. How many times farther away is the average person's nearest neighbor in LA than in San Francisco? The answer is not 1.5.

$\checkmark$ $\star$

1-m3 In Europe, a piece of paper of the standard size, called A4, is a little narrower and taller than its American counterpart. The ratio of the height to the width is the square root of 2, and this has some useful properties. For instance, if you cut an A4 sheet from left to right, you get two smaller sheets that have the same proportions. You can even buy sheets of this smaller size, and they're called A5. There is a whole series of sizes related in this way, all with the same proportions. (a) Compare an A5 sheet to an A4 in terms of area and linear size. (b) The series of paper sizes starts from an A0 sheet, which has an area of one square meter. Suppose we had a series of boxes defined in a similar way: the B0 box has a volume of one cubic meter, two B1 boxes fit exactly inside an B0 box, and so on. What would be the dimensions of a B0 box?

$\checkmark$ $\star$

1-p1 Estimate the number of jellybeans in the figure.

▷ Solution, p. 199

1-p2 Suppose you took enough water out of the oceans to reduce sea level by 1 mm, and you took that water and used it to fill up water bottles. Make an order-of-magnitude estimate of how many water bottles could you fill.

1-p3 If you filled up a small classroom with pennies, about much money would be in the room?

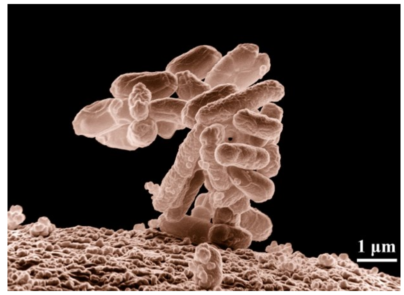
Problem 1-p1.

1-p4 Estimate the mass of one of the hairs in Albert Einstein's moustache, in units of kg.

1-p5 Estimate the number of blades of grass on a football field.

1-p6 Suppose someone built a gigantic apartment building, measuring 10 km $\times$ 10 km at the base. Estimate how tall the building would have to be to have space in it for the entire world's population to live.

1-p7 (a) Using the microscope photo in the figure, estimate the mass of a one cell of the *E. coli* bacterium, which is one of the most common ones in the human intestine. Note the scale at the lower right corner, which is 1 μm. Each of the tubular objects in the column is one cell. (b) The feces in the human intestine are mostly bacteria (some dead, some alive), of which *E. coli* is a large and typical component. Estimate the number of bacteria in your intestines, and compare with the number of human cells in your body, which is believed to be roughly on the order of 10^{13}. (c) Interpreting your result from part b, what does this tell you about the size of a typical human cell compared to the size of a typical bacterial cell?



Problem 1-p7.

1-q1 Estimate the number of man-hours required for building the Great Wall of China.

▷ Solution, p. 200 ⋆

1-q2 Plutonium-239 is one of a small number of important long-lived forms of high-level radioactive nuclear waste. The world's waste stockpiles have about 10^3 metric tons of plutonium. Drinking water is considered safe by U.S. government standards if it contains less than 2×10^{-13} g/cm^3 of plutonium. The amount of radioactivity to which you were exposed by drinking such water on a daily basis would be very small compared to the natural background radiation that you are exposed to every year. Suppose that the world's inventory of plutonium-239 were ground up into an extremely fine dust and then dispersed over the world's oceans, thereby becoming mixed uniformly into the world's water supplies over time. Estimate the resulting concentration of plutonium, and compare with the government standard.

⋆

2 Kinematics in one dimension

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2.1 Velocity

The motion of a particle in one dimension can be described using the function $x(t)$ that gives its position at any time. Its *velocity* is defined by the derivative

$$v = \frac{\mathrm{d}x}{\mathrm{d}t}. \tag{2.1}$$

From the definition, we see that the SI units of velocity are meters per second, m/s. Positive and negative signs indicate the direction of motion, relative to the direction that is arbitrarily called positive when we pick our coordinate system. In the case of constant velocity, we have

$$v = \frac{\Delta x}{\Delta t}, \tag{2.2}$$

where the notation Δ (Greek uppercase "delta," like Latin "D") means "change in," or "final minus initial." When the velocity is not constant, this equation is false, although the quantity $\Delta x/\Delta t$ can be interpreted as a kind of average velocity.

Velocity can only be defined if we choose some arbitrary reference point that we consider to be at rest. Therefore velocity is relative, not absolute. A person aboard a cruising passenger jet might consider the cabin to be at rest, but someone on the ground might say that the plane was moving very fast — relative to the dirt.

To convert velocities from one *frame of reference* to another, we add a constant. If, for example, v_{AB} is the velocity of A relative to B, then

$$v_{AC} = v_{AB} + v_{BC}. \tag{2.3}$$

The *principle of inertia* states that if an object is not acted on by a force, its velocity remains constant. For example, if a rolling soccer ball slows down, the change in its velocity is not because the ball naturally "wants" to slow down but because of a frictional force that the grass exerts on it.

A frame of reference in which the principle of inertia holds is called an *inertial* frame of reference. The earth's surface defines a very nearly inertial frame of reference, but so does the cabin of a cruising passenger jet. Any frame of reference moving at constant velocity, in a straight line, relative to an inertial frame is also an inertial frame. An example of a noninertial frame of reference is a car in an amusement park ride that maneuvers violently.

2.2 Acceleration

The *acceleration* of a particle is defined as the time derivative of the velocity, or the second derivative of the position with respect to time:

$$a = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}^2 x}{\mathrm{d}t^2}. \tag{2.4}$$

It measures the rate at which the velocity is changing. Its units are m/s/s, more commonly written as m/s^2.

Unlike velocity, acceleration is not just a matter of opinion. Observers in different inertial frames of reference agree on accelerations. An acceleration is caused by the force that one object exerts on another.

In the case of constant acceleration, simple al-

gebra and calculus give the following relations:

$$a = \frac{\Delta v}{\Delta t} \tag{2.5}$$

$$x = x_0 + v_0 t + \frac{1}{2}at^2 \tag{2.6}$$

$$v_f^2 = v_0^2 + 2a\Delta x, \tag{2.7}$$

where the subscript 0 (read "nought") means initial, or $t = 0$, and f means final.

Free fall

Galileo showed by experiment that when the only force acting on an object is gravity, the object's acceleration has a value that is independent of the object's mass. This is because the greater force of gravity on a heavier object is exactly compensated for by the object's greater *inertia*, meaning its tendency to resist a change in its motion. For example, if you stand up now and drop a coin side by side with your shoe, you should see them hit the ground at almost the same time, despite the huge disparity in mass. The magnitude of the acceleration of falling objects is notated g, and near the earth's surface g is approximately 9.8 m/s^2. This number is a measure of the strength of the earth's gravitational field.

The inclined plane

If an object slides frictionlessly on a ramp that forms an angle θ with the horizontal, then its acceleration equals $g \sin\theta$. This can be shown based on a looser, generalized statement of the principle of inertia, which leads to the conclusion that the gain in speed on a slope depends only on the vertical drop.[1] For $\theta = 90°$, we recover the case of free fall.

[1] For details of this argument, see Crowell, Mechanics, `lightandmatter.com`, section 3.6.

Graphs of position, velocity, and acceleration

The motion of an object can be represented visually by a stack of graphs of x versus t, v versus t, and a versus t. Figure **??** shows two examples. The slope of the tangent line at a given point on one graph equals the value of the function at the same time in the graph below.

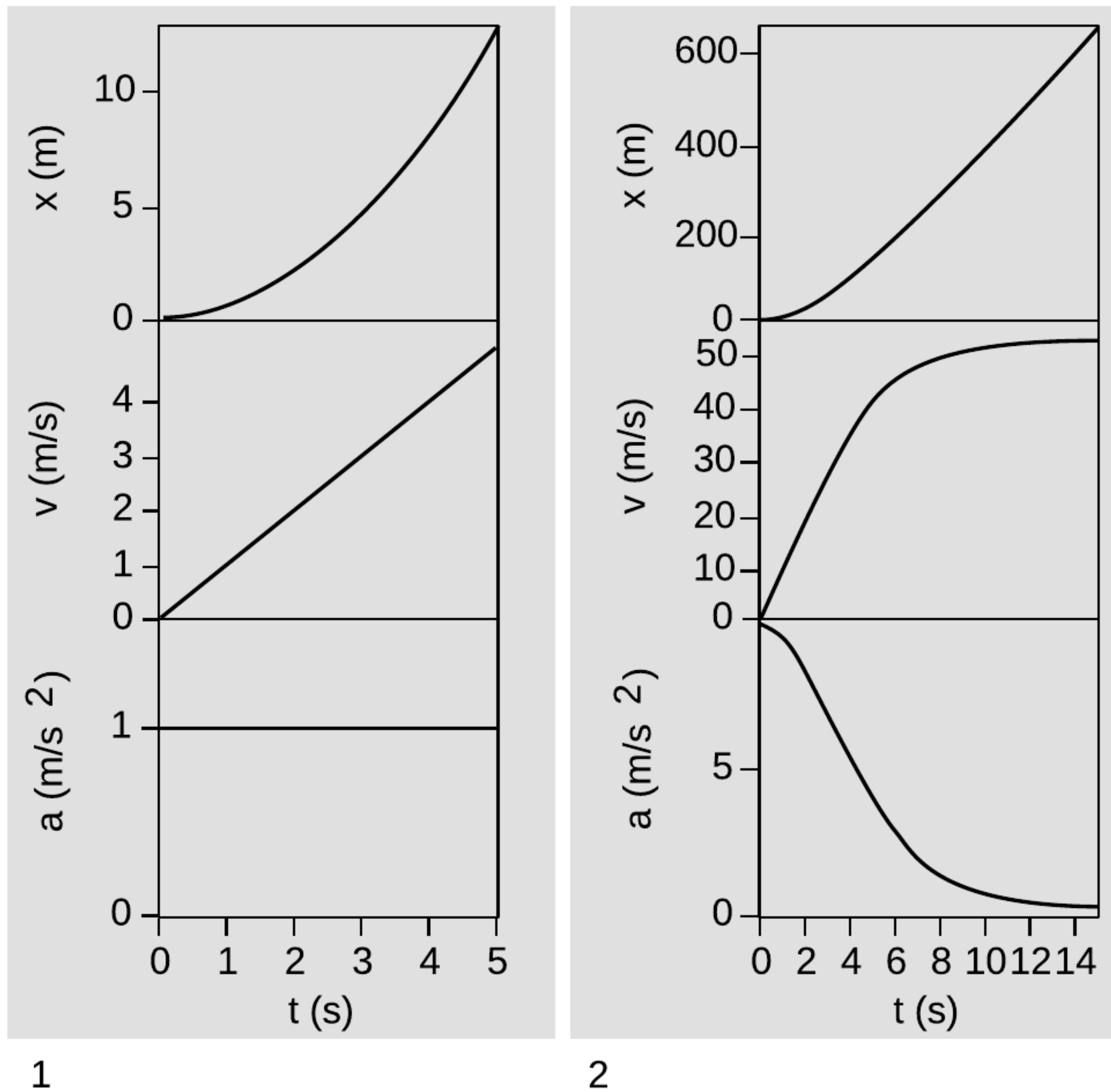


Figure 2.1: 1. Graphs representing the motion of an object moving with a constant acceleration of 1 m/s^2. 2. Graphs for a parachute jumper who initially accelerates at g, but later accelerates more slowly due to air resistance.

Problems

2-a1 You're standing in a freight train, and have no way to see out. If you have to lean to stay on your feet, what, if anything, does that tell you about the train's velocity? Explain.

2-a2 Interpret the general rule $v_{AB} = -v_{BA}$ in words.

2-a3 Wa-Chuen slips away from her father at the mall and walks up the down escalator, so that she stays in one place. Write this in terms of symbols, using the notation with two subscripts introduced in section 2.1.

2-a4 Driving along in your car, you take your foot off the gas, and your speedometer shows a reduction in speed. Describe an inertial frame of reference in which your car was *speeding up* during that same period of time. ⋆

2-b1 (a) Translate the following information into symbols, using the notation with two subscripts introduced in section 2.1. Eowyn is riding on her horse at a velocity of 11 m/s. She twists around in her saddle and fires an arrow backward. Her bow fires arrows at 25 m/s. (b) Find the velocity of the arrow relative to the ground.

2-b2 An airport has a moving walkway to help people move across and/or between terminals quickly. Suppose that you're walking north on such a walkway, where the walkway has speed 3.0 m/s relative to the airport, and you walk at a speed of 2.0 m/s. You pass by your friend, who is off the walkway, traveling south at 1.5 m/s.
(a) What is the magnitude of your velocity with respect to the moving walkway? ✓
(b) What is the magnitude of your velocity with respect to the airport? ✓
(c) What is the magnitude of your velocity with respect to your friend? ✓
(d) If it takes you 45 seconds to get across the airport terminal, how long does it take your friend? ✓

2-b3 On a 20 km bike ride, you ride the first 10 km at an average speed of 8 km/hour. What average speed must you have over the next 10 km if your average speed for the whole ride is to be 12 km/hour? ✓

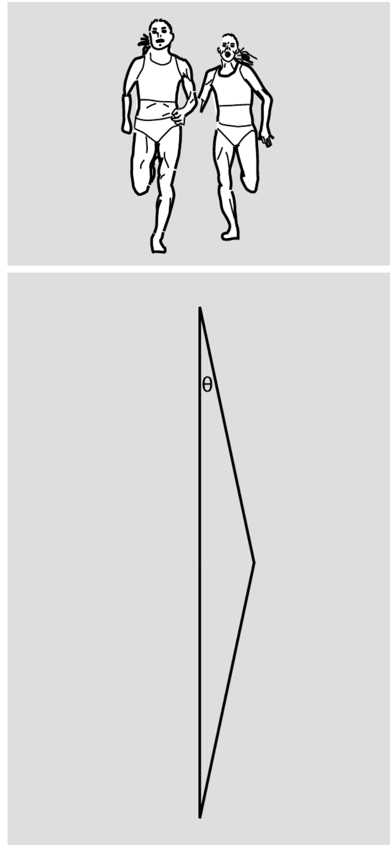
2-b4 (a) In a race, you run the first half of the distance at speed u, and the second half at speed v. Find the over-all speed, i.e., the total distance divided by the total time. ✓
(b) Check the units of your equation .
(c) Check that your answer makes sense in the case where $u = v$.
(d) Show that the dependence of your result on u and v makes sense. That is, first check whether making u bigger makes the result bigger, or smaller. Then compare this with what you expect physically.

2-b5 An object starts moving at $t = 0$, and its position is given by $x = At^5 - Bt^2$, where t is in seconds and x is in meters. A is a non-zero constant. (a) Infer the units of A and B.
(b) Find the velocity as a function of t. ✓
(c) What is the average velocity from 0 to t as a function of time? ✓
(d) At what time t ($t > 0$) is the velocity at t equal to the average velocity from 0 to t? ✓

2-b6 (a) Let R be the radius of the Earth and T the time (one day) that it takes for one rotation. Find the speed at which a point on the equator moves due to the rotation of the earth. ✓
(b) Check the units of your equation .
(c) Check that your answer to part a makes sense in the case where the Earth stops rotating completely, so that T is infinitely long.
(d) Nairobi, Kenya, is very close to the equator. Plugging in numbers to your answer from part a, find Nairobi's speed in meters per second. See the table in the back of the book for the relevant data. For comparison, the speed of sound is about 340 m/s. ✓

2-c1 In running races at distances of 800 meters and longer, runners do not have their own lanes, so in order to pass, they have to go around their opponents. Suppose we adopt the simplified geometrical model suggested by the figure, in which the two runners take equal times to trace out the sides of an isoceles triangle, deviating from parallelism by the angle θ. The runner going straight runs at speed v, while the one who is passing must run at a greater speed. Let the difference in speeds be Δv.
(a) Find Δv in terms of v and θ. $\checkmark$
(b) Check the units of your equation .
(c) Check that your answer makes sense in the special case where $\theta = 0$, i.e., in the case where the runners are on an extremely long straightaway.
(d) Suppose that $\theta = 1.0$ degrees, which is about the smallest value that will allow a runner to pass in the distance available on the straightaway of a track, and let $v = 7.06$ m/s, which is the women's world record pace at 800 meters. Plug numbers into your equation from part a to determine Δv, and comment on the result. $\checkmark$

$\star$

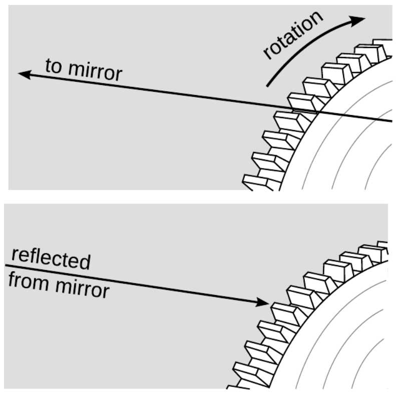


Problem 2-c1.

2-c2 In 1849, Fizeau carried out the first terrestrial measurement of the speed of light; previous measurements by Roemer and Bradley had involved astronomical observation. The figure shows a simplified conceptual representation of Fizeau's experiment. A ray of light from a bright source was directed through the teeth at the edge of a spinning cogwheel. After traveling a distance L, it was reflected from a mirror and returned along the same path. The figure shows the case in which the ray passes between two teeth, but when it returns, the wheel has rotated by half the spacing of the teeth, so that the ray is blocked. When this condition is achieved, the observer looking through the teeth toward the far-off mirror sees it go completely dark. Fizeau adjusted the speed of the wheel to achieve this condition and recorded the rate of rotation to be f rotations per second. Let the number of teeth on the wheel be n.

(a) Find the speed of light c in terms of L, n, and f. $\checkmark$
(b) Check the units of your equation . (Here f's units of rotations per second should be taken as inverse seconds, s^{-1}, since the number of rotations in a second is a unitless count.)
(c) Imagine that you are Fizeau trying to design this experiment. The speed of light is a huge number in ordinary units. Use your equation from part a to determine whether increasing c requires an increase in L, or a decrease. Do the same for n and f. Based on this, decide for each of these variables whether you want a value that is as big as possible, or as small as possible.
(d) Fizeau used $L = 8633$ m, $f = 12.6\ \text{s}^{-1}$, and $n = 720$. Plug in to your equation from part

a and extract the speed of light from his data. ✓



Problem 2-c2.

2-e1 What is the acceleration of a car that moves at a steady velocity of 100 km/h for 100 seconds? Explain your answer. [Based on a problem by Hewitt.]

2-e2 Alice drops a rock off a cliff. Bubba shoots a gun straight down from the edge of the same cliff. Compare the accelerations of the rock and the bullet while they are in the air on the way down. [Based on a problem by Serway and Faughn.]

2-e3 A toy car is released on one side of a piece of track that is bent into an upright U shape. The car goes back and forth. When the car reaches the limit of its motion on one side, its velocity is zero. Is its acceleration also zero? Explain using a $v-t$ graph. [Based on a problem by Serway and Faughn.]

2-e4 If an object starts accelerating from rest, we have $v^2 = 2a\Delta x$ for its speed after it has traveled a distance Δx. Explain in words why it makes sense that the equation has velocity squared, but distance only to the first power. Don't recapitulate the derivation in the book, or give a justification based on units. The point is to explain what this feature of the equation tells us about the way speed increases as more distance is covered. ⋆

2-f1 On New Year's Eve, a stupid person fires a pistol straight up. The bullet leaves the gun at a speed of 100 m/s. How long does it take before the bullet hits the ground?

2-f2 A physics homework question asks, "If you start from rest and accelerate at 1.54 m/s^2 for 3.29 s, how far do you travel by the end of that time?" A student answers as follows:

$$1.54 \times 3.29 = 5.07 \text{ m}$$

His Aunt Wanda is good with numbers, but has never taken physics. She doesn't know the formula for the distance traveled under constant acceleration over a given amount of time, but she tells her nephew his answer cannot be right. How does she know?

2-f3 You are looking into a deep well. It is dark, and you cannot see the bottom. You want to find out how deep it is, so you drop a rock in, and you hear a splash 3.0 seconds later. How deep is the well? ✓

2-f4 Consider the following passage from Alice in Wonderland, in which Alice has been falling for a long time down a rabbit hole:

Down, down, down. Would the fall *never* come to an end? "I wonder how many miles I've fallen by this time?" she said aloud. "I must be getting somewhere near the center of the earth. Let me see: that would be four thousand miles down, I think" (for, you see, Alice had learned several things of this sort in her lessons in the schoolroom, and though this was not a *very* good opportunity for showing off her knowledge, as there was no one to listen to her, still it was good practice to say it over)...

Alice doesn't know much physics, but let's try to calculate the amount of time it would take to fall four thousand miles, starting from rest with an acceleration of 10 m/s^2. This is really only a

lower limit; if there really was a hole that deep, the fall would actually take a longer time than the one you calculate, both because there is air friction and because gravity gets weaker as you get deeper (at the center of the earth, g is zero, because the earth is pulling you equally in every direction at once).

√

2-f5 You shove a box with initial velocity 2.0 m/s, and it stops after sliding 1.3 m. What is the magnitude of the deceleration, assuming it is constant?

√

2-f6 You're an astronaut, and you've arrived on planet X, which is airless. You drop a hammer from a height of 1.00 m and find that it takes 350 ms to fall to the ground. What is the acceleration due to gravity on planet X?

√

2-i1 Mr. Whiskers the cat can jump 2.0 meters vertically (undergoing free-fall while in the air). (a) What initial velocity must he have in order to jump this high? √
(b) How long does it take him to reach his maximum height from the moment he leaves the ground? √
(c) From the start of the jump to the time when he lands on the ground again, how long is he in the air?

√

2-i2 A baseball pitcher throws a fastball. Her hand accelerates the ball from rest to 45.0 m/s over a distance 1.5 meters. For the purposes of this problem, we will make the simplifying assumption that this acceleration is constant. (a) What is the ball's acceleration? √
(b) How much time does it take for the pitcher to accelerate the ball? √
(c) If home plate is 18.0 meters away from where the pitcher releases the baseball, how much total time does the baseball take to get there, assuming it moves with constant velocity as soon as it leaves the pitcher's hand? Include both the time required for acceleration and the time the ball spends on the fly.

√

2-i3 The photo shows Apollo 16 astronaut John Young jumping on the moon and saluting at the top of his jump. The video footage of the jump shows him staying aloft for 1.45 seconds. Gravity on the moon is 1/6 as strong as on the earth. Compute the height of the jump.

√

Problem 2-i3.

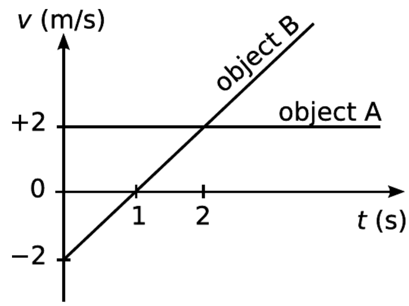
2-i4 Find the error in the following calculation. A student wants to find the distance traveled by a car that accelerates from rest for 5.0 s with an acceleration of 2.0 m/s^2. First he solves $a = \Delta v/\Delta t$ for $\Delta v = 10$ m/s. Then he multiplies to find (10 m/s)(5.0 s) = 50 m. Do not just recalculate the result by a different method; if that was all you did, you'd have no way of knowing which calculation was correct, yours or his.

2-i5 A naughty child drops a golf ball from the roof of your apartment building, and you see it drop past your window. It takes the ball time T to traverse the window's height H. Find the initial speed of the ball when it first came into view.

√

2-i6 Objects A and B, with $v(t)$ graphs shown in the figure, both leave the origin at time $t = 0$ s. When do they cross paths again?

√



Problem 2-i6.

2-i7 An elevator is moving upward at constant speed of 2.50 m/s. A bolt in the elevator's ceiling, 3.00 m above the floor, works loose and falls. (a) How long does it take for the bolt to fall to the floor? ✓
(b) What is the speed of the bolt just as it hits the floor, according to an observer in the elevator? ✓
(c) What is the speed of the bolt just as it hits the elevator's floor, according to an observer standing on one of the floor landings of the building? ✓

2-i8 You're in your Honda, cruising on the freeway at velocity u, when, up ahead at distance L, a Ford pickup truck cuts in front of you while moving at constant velocity $u/2$. Like half the speed that any reasonable person would go! Let positive x be the direction of motion, and let your position be $x = 0$ at $t = 0$. To avoid a collision, you immediately slam on the brakes and start decelerating with acceleration $-a$, where a is a positive constant.
(a) Write an equation for $x_{\mathrm{F}}(t)$, the position of the Ford as a function of time, as they trundle onward obliviously. ✓
(b) Write an equation for $x_{\mathrm{H}}(t)$, the position of your Honda as a function of time. ✓
(c) By subtracting one from the other, find an expression for the distance between the two vehicles as a function of time, $d(t)$. ✓
(d) Find the minimum value of a that avoids a collision. ✓
(e) Show that your answer to part e has units that make sense.
(f) Show that the dependence of your answer on the variables makes sense physically.

2-i9 You're in your Honda on the freeway traveling behind a Ford pickup truck. The truck is moving at a steady speed of 30.0 m/s, you're speeding at 40.0 m/s, and you're cruising 45 meters behind the Ford. At $t = 0$, the Ford slams on his/her brakes, and decelerates at a rate of 5.0 $\mathrm{m/s^2}$. You don't notice this until $t = 1.0$ s, where you begin decelerating at 10.0 $\mathrm{m/s^2}$. Let positive x be the direction of motion, and let your position be $x = 0$ at $t = 0$. The goal is to find the motion of each vehicle and determine whether there is a collision.
(a) Doing this entire calculation purely numerically would be very cumbersome, and it would be difficult to tell whether you had made mistakes. Translate the given information into algebra symbols, and find an equation for $x_{\mathrm{F}}(t)$, the position of the Ford as a function of time. ✓
(b) Write a similar symbolic equation for $x_{\mathrm{H}}(t)$ (for $t > 1$ s), the position of the Honda as a function of time. Why isn't this formula valid for $t < 1$ s? ✓
(c) By subtracting one from the other, find an expression for the distance between the two vehicles as a function of time, $d(t)$ (valid for $t > 1$ s until the truck stops). Does the equation $d(t) = 0$ have any solutions? What does this tell you? ✓
(d) Because this is a fairly complicated calculation, we will find the result in two different ways and check them against each other. Plug numbers back in to the results of parts a and b, replacing the symbols in the constant coefficients, and graph the two functions using a graphing calculator or an online utility such as desmos.com.
(e) As you should have discovered in parts c and d, the two vehicles do not collide. At what time does the minimum distance occur, and what is that distance?
✓

2-i10 You're standing on the roof of your science building, which is 10.0 meters above the ground. You have a rock in your hand, which you can throw with a maximum speed of 10.0 m/s.
(a) How long would it take for the rock to hit the ground if you released the rock from rest? $\surd$
(b) How long would it take for the rock to hit the ground if you threw the rock straight downward? $\surd$
(c) How long would it take for the rock to hit the ground if you threw the rock straight upward? $\surd$
(d) If you threw the rock straight upward, how high would it get above the ground? $\surd$

2-i11 You're standing on the roof of your science building. You drop a rock from rest and notice that it takes an amount of time T to hit the ground. Express your answers to the following questions in terms of T and the acceleration due to gravity, g.
(a) How high is the building? $\surd$
(b) How fast must you throw the rock straight up if the rock is to take $2T$ to hit the ground? $\surd$
(c) For the situation described in part b, how long does it take from the time you let the rock go to when the rock reaches maximum height? $\surd$
(d) For the same situation, what is the maximum height that the rock gets to above the ground? $\surd$

2-i12 You take a trip in your spaceship to another star. Setting off, you increase your speed at a constant acceleration. Once you get halfway there, you start decelerating, at the same rate, so that by the time you get there, you have slowed down to zero speed. You see the tourist attractions, and then head home by the same method.
(a) Find a formula for the time, T, required for the round trip, in terms of d, the distance from our sun to the star, and a, the magnitude of the acceleration. Note that the acceleration is not constant over the whole trip, but the trip can be broken up into constant-acceleration parts.
(b) The nearest star to the Earth (other than our own sun) is Proxima Centauri, at a distance of $d = 4 \times 10^{16}$ m. Suppose you use an acceleration of $a = 10$ m/s^2, just enough to compensate for the lack of true gravity and make you feel comfortable. How long does the round trip take, in years?
(c) Using the same numbers for d and a, find your maximum speed. Compare this to the speed of light, which is 3.0×10^8 m/s. (Later in this course, you will learn that there are some new things going on in physics when one gets close to the speed of light, and that it is impossible to exceed the speed of light. For now, though, just use the simpler ideas you've learned so far.) $\surd$

2-k1 If the acceleration of gravity on Mars is 1/3 that on Earth, how many times longer does it take for a rock to drop the same distance on Mars? Ignore air resistance.
▷ Solution, p. 200

2-k2 Starting from rest, a ball rolls down a ramp, traveling a distance L and picking up a final speed v. How much of the distance did the ball have to cover before achieving a speed of $v/2$? [Based on a problem by Arnold Arons.] $\star$

2-k3 Suppose you can hit a tennis ball vertically upward with a certain initial speed, independent of what planet you're on.
(a) If the height the ball reaches on Earth is H, what is the height the ball will reach on Pluto, where the acceleration due to gravity is about 1/15th the value on Earth? $\surd$
(b) If the amount of time the ball spends in the air on Earth is 3.0 seconds, how long would it spend in the air on Pluto? $\surd$

2-k4 You climb half-way up a tree, and drop a rock. Then you climb to the top, and drop another rock. How many times greater is the velocity of the second rock on impact? Explain. (The answer is not two times greater.)

2-k5 Most people don't know that *Spinosaurus aegyptiacus*, not *Tyrannosaurus rex*, was the biggest theropod dinosaur. We can't put a dinosaur on a track and time it

in the 100 meter dash, so we can only infer from physical models how fast it could have run. When an animal walks at a normal pace, typically its legs swing more or less like pendulums of the same length ℓ. As a further simplification of this model, let's imagine that the leg simply moves at a fixed acceleration as it falls to the ground. That is, we model the time for a quarter of a stride cycle as being the same as the time required for free fall from a height ℓ. *S. aegyptiacus* had legs about four times longer than those of a human. (a) Compare the time required for a human's stride cycle to that for *S. aegyptiacus*. ✓
(b) Compare their running speeds.
✓

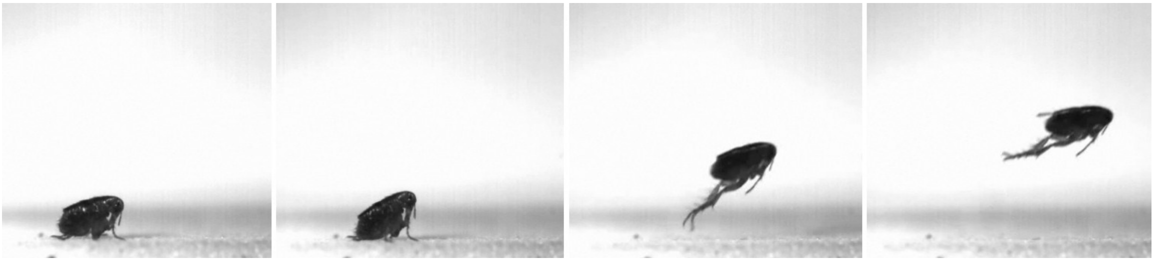
2-k6 Engineering professor Qingming Li used sensors and video cameras to study punches delivered in the lab by British former welterweight boxing champion Ricky "the Hitman" Hatton. For comparison, Li also let a TV sports reporter put on the gloves and throw punches. The time it took for Hatton's best punch to arrive, i.e., the time his opponent would have had to react, was about 0.47 of that for the reporter. Let's assume that the fist starts from rest and moves with constant acceleration all the way up until impact, at some fixed distance (arm's length). Compare Hatton's acceleration to the reporter's.
✓

2-k7 Aircraft carriers originated in World War I, and the first landing on a carrier was performed by E.H. Dunning in a Sopwith Pup biplane, landing on HMS Furious. (Dunning was killed the second time he attempted the feat.) In such a landing, the pilot slows down to just above the plane's stall speed, which is the minimum speed at which the plane can fly without stalling. The plane then lands and is caught by cables and decelerated as it travels the length of the flight deck. Comparing a modern US F-14 fighter jet landing on an Enterprise-class carrier to Dunning's original exploit, the stall speed is greater by a factor of 4.8, and to accomodate this, the length of the flight deck is greater by a factor of 1.9. Which deceleration is greater, and by what factor?
✓

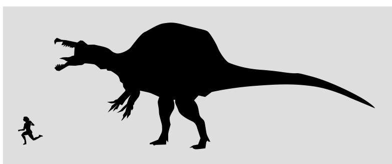
2-k8 In college-level women's softball in the U.S., typically a pitcher is expected to be at least 1.75 m tall, but Virginia Tech pitcher Jasmin Harrell is 1.62 m. Although a pitcher actually throws by stepping forward and swinging her arm in a circle, let's make a simplified physical model to estimate how much of a disadvantage Harrell has had to overcome due to her height. We'll pretend that the pitcher gives the ball a constant acceleration in a straight line, and that the length of this line is proportional to the pitcher's height. Compare the acceleration Harrell would have to supply with the acceleration that would suffice for a pitcher of the nominal minimum height, if both were to throw a pitch at the same speed.
✓

2-k9 When the police engage in a high-speed chase on city streets, it can be extremely dangerous both to the police and to other motorists and pedestrians. Suppose that the police car must travel at a speed that is limited by the need to be able to stop before hitting a baby carriage, and that the distance at which the driver first sees the baby carriage is fixed. Tests show that in a panic stop from high speed, a police car based on a Chevy Impala has a deceleration 9% greater than that of a Dodge Intrepid. Compare the maximum safe speeds for the two cars.
✓

2-n1 The figure shows a practical, simple experiment for determining g to high precision. Two steel balls are suspended from electromagnets, and are released simultaneously when the electric current is shut off. They fall through unequal heights Δx_1 and Δx_2. A computer records the sounds through a microphone as first one ball and then the other strikes the floor. From this recording, we can accurately determine the quantity T defined as $T = \Delta t_2 - \Delta t_1$, i.e., the time lag between the first and second impacts. Note that since the balls do not make any sound



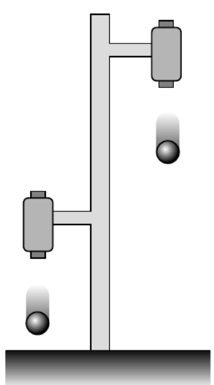
Problem 2-n2.



Problem 2-k5.

when they are released, we have no way of measuring the individual times Δt_2 and Δt_1.
(a) Find an equation for g in terms of the measured quantities T, Δx_1 and Δx_2. ✓
(b) Check the units of your equation.
(c) Check that your equation gives the correct result in the case where Δx_1 is very close to zero. However, is this case realistic?
(d) What happens when $\Delta x_1 = \Delta x_2$? Discuss this both mathematically and physically.

⋆



Problem 2-n1.

2-n2 Some fleas can jump as high as 30 cm. The flea only has a short time to build up speed — the time during which its center of mass is accelerating upward but its feet are still in contact with the ground. Make an order-of-magnitude estimate of the acceleration the flea needs to have while straightening its legs, and state your answer in units of g, i.e., how many "g's it pulls." (For comparison, fighter pilots black out or die if they exceed about 5 or 10 g's.)

⋆

2-n3 The speed required for a low-earth orbit is 7.9×10^3 m/s. When a rocket is launched into orbit, it goes up a little at first to get above almost all of the atmosphere, but then tips over horizontally to build up to orbital speed. Suppose the horizontal acceleration is limited to $3g$ to keep from damaging the cargo (or hurting the crew, for a crewed flight). (a) What is the minimum distance the rocket must travel downrange before it reaches orbital speed? How much does it matter whether you take into account the initial eastward velocity due to the rotation of the earth? (b) Rather than a rocket ship, it might be advantageous to use a railgun design, in which the craft would be accelerated to orbital speeds along a railroad track. This has the advantage that it isn't necessary to lift a large mass of fuel, since the energy source is external. Based on your answer to part a, comment on the feasibility of this design for crewed launches from the earth's surface.

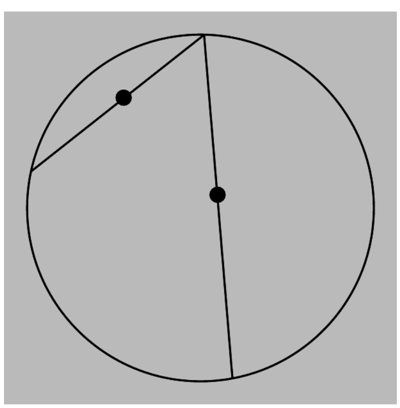
⋆

2-n4 When an object slides frictionlessly down a plane inclined at an angle θ, its acceleration equals $g \sin\theta$ (p. 16). Suppose that a person on a bike is to coast down a ramp, starting from rest, and then coast back up an identical ramp, tracing a "V." The horizontal distance is fixed to be $2w$, and we want to set the depth of the "V" so as to achieve the minimal possible value t^* for the total time.
(a) Based only on units, infer the form of the expression for t^* in terms of w, up to a unitless multiplicative constant.
(b) Find the angle that minimizes the time.
(c) Complete the determination of t^* by finding the unitless constant.

▷ Solution, p. 200 ⋆

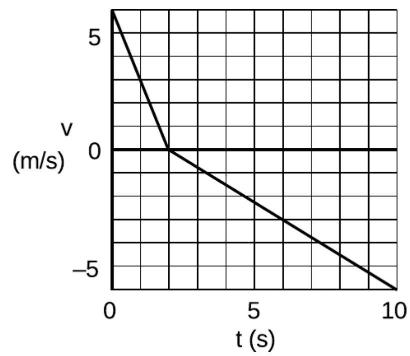
2-n5 The figure shows a circle in a vertical plane, with two wires positioned along chords of the circle. The top of each wire coincides with the top of the circle. Beads slide frictionlessly on the wires. If the beads are released simultaneously at the top, which one wins the race? You will need the fact that the acceleration equals $g \sin\theta$ (p. 16).

⋆



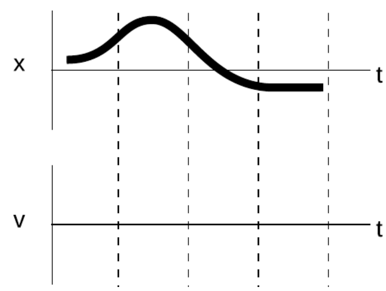
Problem 2-n5.

2-o1 The graph represents the motion of a ball that rolls up a hill and then back down. When does the ball return to the location it had at $t = 0$?



Problem 2-o1.

2-o2 The top part of the figure shows the position-versus-time graph for an object moving in one dimension. On the bottom part of the figure, sketch the corresponding v-versus-t graph.



Problem 2-o2.

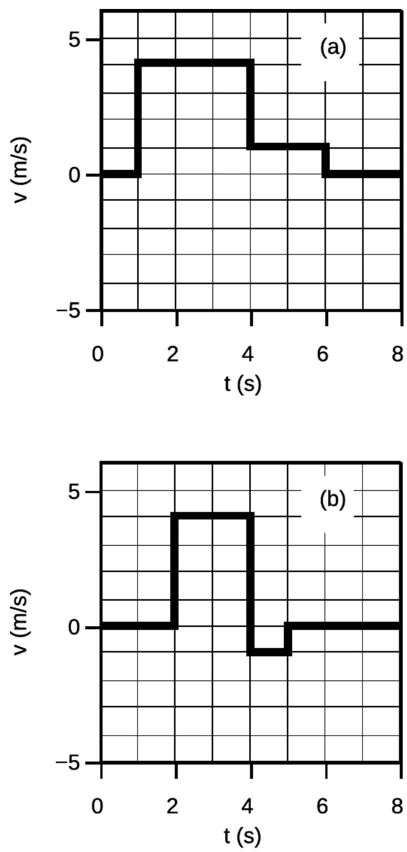
2-o3 For each of the two graphs, find the change in position Δx from beginning to end.

✓

2-o4 The graph represents the velocity of a bee along a straight line. At $t = 0$, the bee is at the hive. (a) When is the bee farthest from the hive? (b) How far is the bee at its farthest point from the hive? (c) At $t = 13$ s, how far is the bee from the hive?

✓

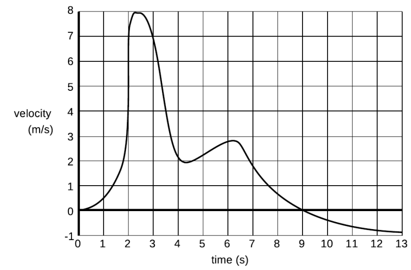
2-o5 Decide whether the following statements about one-dimensional motion are true or false:
(a) The area under a $v(t)$ curve gives the acceleration of the object.



Problem 2-o3.

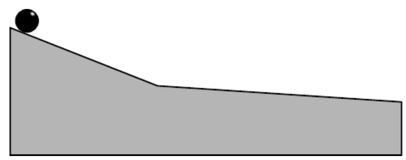
(b) The area under a $a(t)$ curve gives the change in velocity of the object.
(c) The slope of a $a(t)$ curve at time T is the value $v(T)$.
(d) The slope of the $v(t)$ curve at time T is the value $a(T)$.
(e) The displacement Δx from t_1 to t_2 is equal to the area under the $a(t)$ curve from t_1 to t_2.

2-o6 (a) The ball is released at the top of the ramp shown in the figure. Friction is negligible. Use physical reasoning to draw $v-t$ and $a-t$ graphs. Assume that the ball doesn't bounce at the point where the ramp changes slope. (b) Do the same for the case where the ball is rolled up



Problem 2-o4.

the slope from the right side, but doesn't quite have enough speed to make it over the top.


Problem 2-o6.

2-o7 You throw a rubber ball up, and it falls and bounces several times. Draw graphs of position, velocity, and acceleration as functions of time.

2-p1 (a) Express the chain rule in Leibniz ("d") notation, and show that it always results in an answer whose units make sense.
(b) An object has a position as a function of time given by $x = A\cos(bt)$, where A and b are constants. Infer the units of A and b, and interpret their physical meanings.
(c) Find the velocity of this object, and check that the chain rule has indeed given an answer with the right units.

2-p2 In July 1999, Popular Mechanics carried out tests to find which car sold by a major auto maker could cover a quarter mile (402 meters) in the shortest time, starting from rest. Because the distance is so short, this type of test is designed mainly to favor the car with the greatest

acceleration, not the greatest maximum speed (which is irrelevant to the average person). The winner was the Dodge Viper, with a time of 12.08 s. The car's top (and presumably final) speed was 118.51 miles per hour (52.98 m/s). (a) If a car, starting from rest and moving with *constant* acceleration, covers a quarter mile in this time interval, what is its acceleration? (b) What would be the final speed of a car that covered a quarter mile with the constant acceleration you found in part a? (c) Based on the discrepancy between your answer in part b and the actual final speed of the Viper, what do you conclude about how its acceleration changed over time?

▷ Solution, p. 200 ⋆

2-p3 A honeybee's position as a function of time is given by $x = 10t - t^3$, where t is in seconds and x in meters. What is its velocity at $t = 3.0$ s?

✓

2-p4 Objects A and B move along the x axis. The acceleration of both objects as functions of time is given by $a(t) = (3.00\ \text{m/s}^3)t$. Object A starts (at $t = 0$) from rest at the origin, and object B starts at $x = 5.00$ m, initially moving in the negative x direction with speed 9.00 m/s.
(a) What is A's velocity at time $t = 2.00$ s? ✓
(b) What is A's position at the same time? ✓
(c) What is B's velocity at the same time? ✓
(d) What is B's position at the same time? ✓
(e) Consider a frame of reference in which A is at rest, such as the frame that would naturally be adopted by an observer moving along with A. Describe B's motion in this frame.
(f) After they start, is there any time at which A and B collide?

2-p5 The position of a particle moving on the x-axis is described by the equation $x(t) = t^3 - 4t^2$ (with x in meters and t in seconds). Consider the times $t = -1$, 0, 1, 2, and 3 seconds. For which of these times is the particle slowing down?

2-p6 Freddi Fish$^{(\text{TM})}$ has a position as a function of time given by $x = a/(b + t^2)$. (a) Infer the units of the constants a and b. (b) Find her maximum speed. (c) Check that your answer has the right units.

✓

2-p7 Let t be the time that has elapsed since the Big Bang. In that time, one would imagine that light, traveling at speed c, has been able to travel a maximum distance ct. (In fact the distance is several times more than this, because according to Einstein's theory of general relativity, space itself has been expanding while the ray of light was in transit.) The portion of the universe that we can observe would then be a sphere of radius ct, with volume $v = (4/3)\pi r^3 = (4/3)\pi(ct)^3$. Compute the rate $\mathrm{d}v/\mathrm{d}t$ at which the volume of the observable universe is increasing, and check that your answer has the right units.

✓

2-p8 Sometimes doors are built with mechanisms that automatically close them after they have been opened. The designer can set both the strength of the spring and the amount of friction. If there is too much friction in relation to the strength of the spring, the door takes too long to close, but if there is too little, the door will oscillate. For an optimal design, we get motion of the form $x = cte^{-bt}$, where x is the position of some point on the door, and c and b are positive constants. (Similar systems are used for other mechanical devices, such as stereo speakers and the recoil mechanisms of guns.) In this example, the door moves in the positive direction up until a certain time, then stops and settles back in the negative direction, eventually approaching $x = 0$. This would be the type of motion we would get if someone flung a door open and the door closer then brought it back closed again.
(a) Infer the units of the constants b and c.
(b) Find the door's maximum speed (i.e., the greatest absolute value of its velocity) as it comes back to the closed position. ✓
(c) Show that your answer has units that make sense.

⋆

2-p9 A person is parachute jumping. During the time between when she leaps out of the plane

and when she opens her chute, her altitude is given by an equation of the form

$$y = b - c\left(t + ke^{-t/k}\right),$$

where e is the base of natural logarithms, and b, c, and k are constants. Because of air resistance, her velocity does not increase at a steady rate as it would for an object falling in vacuum.
(a) What units would b, c, and k have to have for the equation to make sense?
(b) Find the person's velocity, v, as a function of time. [You will need to use the chain rule, and the fact that $\mathrm{d}(e^x)/\,\mathrm{d}x = e^x$.] ✓
(c) Use your answer from part (b) to get an interpretation of the constant c. [Hint: e^{-x} approaches zero for large values of x.]
(d) Find the person's acceleration, a, as a function of time. ✓
(e) Use your answer from part (d) to show that if she waits long enough to open her chute, her acceleration will become very small.

⋆

3 Kinematics in three dimensions

This is not a textbook. It's a book of problems meant to be used along with a textbook. Although each chapter of this book starts with a brief summary of the relevant physics, that summary is not meant to be enough to allow the reader to actually learn the subject from scratch. The purpose of the summary is to show what material is needed in order to do the problems, and to show what terminology and notation are being used.

3.1 Vectors

Most of the things we want to measure in physics fall into two categories, called *vectors* and *scalars*. A scalar is something that doesn't change when you turn it around, while a vector does change when you rotate it, and the way in which it changes is the same as the way in which a pointer such as a pencil or an arrow would change. For example, temperature is a scalar: a hot cup of coffee doesn't change its temperature when we turn it around. Force is a vector. When I play tug-of-war with my dog, her force and mine are the same in strength, but they're in opposite directions. If we swap positions, our forces reverse their directions, just as a pair of arrows would.

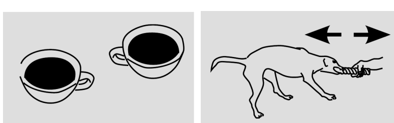


Figure 3.1: Temperature is a scalar. Force is a vector.

To distinguish vectors from scalars, we write them differently, e.g., p for a scalar and boldface $\mathbf{p}$ for a vector. In handwriting, a vector is written with an arrow over it, $\vec{p}$.

Not everything is a scalar or a vector. For example, playing cards are designed in a symmetric way, so that they look the same after a 180-degree rotation. The orientation of the card is not a scalar, because it changes under a rotation, but it's not a vector, because it doesn't behave the way an arrow would under a 180-degree rotation.

In kinematics, the simplest example of a vector is a motion from one place to another, called a displacement vector.

A vector $\mathbf{A}$ has a magnitude $|\mathbf{A}|$, which means its size, length, or amount. Rotating a vector can change the vector, but will never change its magnitude.

Scalars are just numbers, and we do arithmetic on them in the usual way. Vectors are different. Vectors can be added graphically by placing them tip to tail, and then drawing a vector from the tail of the first vector to the tip of the second vector. A vector can be multiplied by a scalar to give a new vector. For instance, if $\mathbf{A}$ is a vector, then $2\mathbf{A}$ is a vector that has the same direction but twice the magnitude. Multiplying by -1 is the same as flipping the vector, $-\mathbf{A} = (-1)\mathbf{A}$. Vector subtraction can be accomplished by flipping and adding.

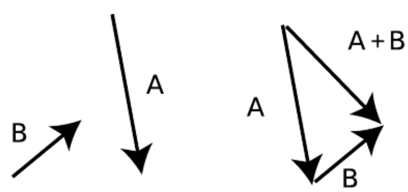


Figure 3.2: Graphical addition of vectors.

Suppose that a sailboat undergoes a displacement $\mathbf{h}$ while moving near a pier. We can define a number called the *component* of $\mathbf{h}$ parallel to the pier, which is the distance the boat has moved along the pier, ignoring an motion toward or away from the pier. If we arbitrarily define one direction along the pier as positive, then the component has a sign.

Often it is convenient to work with compo-

nents of a vector along the coordinate axes. If we pick a Cartesian coordinate system with x, y, and z axes, then any vector can be specified according to its x, y, and z components. We have previously given a graphical definition for vector addition. This is equivalent to adding components.

Unit vector notation

Suppose we want to tell someone that a certain vector $\mathbf{A}$ in two dimensions has components $A_x = 3$ and $A_y = 7$. A more compact way of notating this is $\mathbf{A} = 3\hat{\mathbf{x}} + 7\hat{\mathbf{y}}$, where $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$, read "x-hat" and "y-hat," are the vectors with magnitude one that point in the positive x and y directions. Some authors notate the unit vectors as $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ rather than $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$.

Rotational invariance

Certain vector operations are useful and others are not. Consider the operation of multiplying two vectors component by component to produce a third vector:

$$R_x = P_x Q_x$$
$$R_y = P_y Q_y$$
$$R_z = P_z Q_z.$$

This operation will never be useful in physics because it can give different results depending on our choice of coordinates. That is, if we change our coordinate system by rotating the axes, then the resulting vector $\mathbf{R}$ will of course have different components, but these will not (except in exceptional cases) be the components of the same vector expressed in the new coordinates. We say that this operation is not *rotationally invariant.*

The universe doesn't come equipped with coordinates, so if any vector operation is to be useful in physics, it must be rotationally invariant. Vector addition, for example, is rotationally invariant, since we can define it using tip-to-tail graphical addition, and this definition doesn't even refer to any coordinate system. This rotational invariance would still have held, but might not have been so obvious, if we had defined addition in terms of addition of components.

Dot and cross product

The vector *dot product* $\mathbf{A} \cdot \mathbf{B}$ is defined as the (signed) component of $\mathbf{A}$ parallel to $\mathbf{B}$. It is a scalar. If we know the magnitudes of the vectors and the angle θ_{AB} between them, we can compute the dot product as $|\mathbf{A}||\mathbf{B}|\cos\theta_{AB}$. If we know the components of the vectors in a particular coordinate system, we can express the dot product as $A_xB_x + A_yB_y + A_zB_z$.

The dot product is useful simply as a geometrical tool, but later in this course we will also see that it has physical applications that include mechanical work, as well as many examples in electricity and magnetism, such as electric flux.

There is also a way of multiplying two vectors to obtain a vector result. This is called the vector *cross product*, $\mathbf{C} = \mathbf{A} \times \mathbf{B}$. The magnitude of the cross product is the area of the parallelogram illustrated in figure 3.3. The direction of the cross product is perpendicular to the plane in which $\mathbf{A}$ and $\mathbf{B}$ lie. There are two such directions, and of these two, we choose the one defined by the right-hand rule illustrated in figure 3.4.

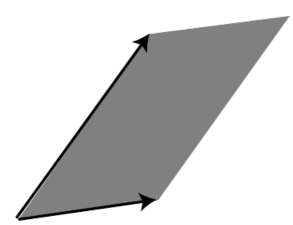


Figure 3.3: The magnitude of the cross product is the area of the shaded parallelogram.

Important physical applications of the cross product include torque, angular momentum, and magnetic forces.

Unlike ordinary multiplication of real numbers, the cross product is anticommutative, $\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$. The magnitude of the cross product can be expressed as $|\mathbf{A}||\mathbf{B}|\sin\theta_{AB}$. In terms

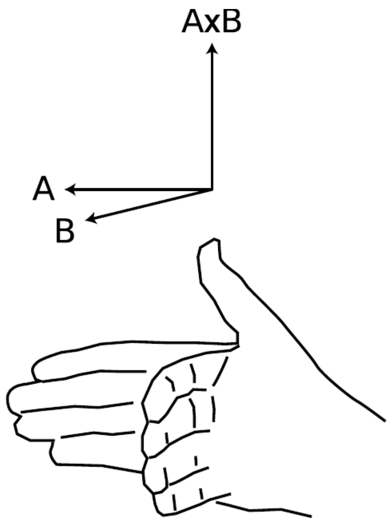


Figure 3.4: The right-hand rule for the direction of the vector cross product.

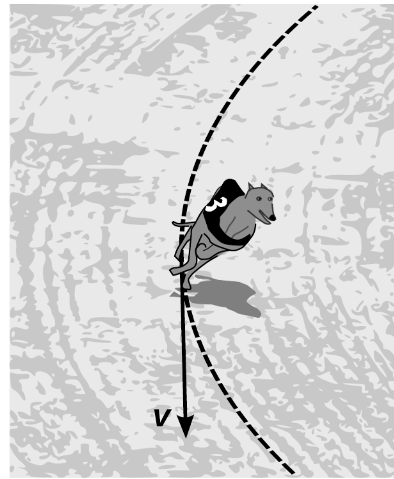


Figure 3.5: The racing greyhound's velocity vector is in the direction of its motion, i.e., tangent to its curved path.

of the components, we have

$$\begin{aligned}\mathbf{C}_x &= A_y B_z - B_y A_z\\ \mathbf{C}_y &= A_z B_x - B_z A_x\\ \mathbf{C}_z &= A_x B_y - B_x A_y.\end{aligned}$$

3.2 Motion

Velocity and acceleration

If an object undergoes an infinitesimal displacement $\mathrm{d}\mathbf{r}$ in an infinitesimal time interval $\mathrm{d}t$, then its velocity vector is the derivative $\mathbf{v} = \mathrm{d}\mathbf{r}/\,\mathrm{d}t$. This type of derivative of a vector can be computed by differentiating each component separately. The acceleration is the second derivative $\mathrm{d}^2\,\mathbf{r}/\,\mathrm{d}t^2$.

The velocity vector has a magnitude that indicates the speed of motion, and a direction that gives the direction of the motion. We saw in section 2.1 that velocities add in relative motion. To generalize this to more than one dimension, we use vector addition.

The acceleration vector does *not* necessarily point in the direction of motion. It points in the direction that an accelerometer would point, as in figure 3.6.

Figure 3.6: The car has just swerved to the right. The air freshener hanging from the rear-view mirror acts as an accelerometer, showing that the acceleration vector is to the right.

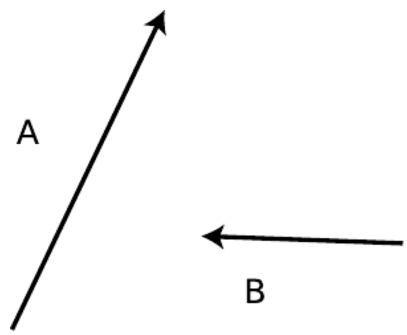
Projectiles and the inclined plane

Forces cause accelerations, not velocities. In particular, the downward force of gravity causes a downward acceleration vector. After a projectile is launched, the only force on it is gravity, so its acceleration vector points straight down. Therefore the horizontal part of its motion has

constant velocity. The vertical and horizontal motions of a projectile are independent. Neither affects the other.

Problems

3-a1 The figure shows vectors **A** and **B**. Graphically calculate the following.
$\mathbf{A}+\mathbf{B}$, $\mathbf{A}-\mathbf{B}$, $\mathbf{B}-\mathbf{A}$, $-2\mathbf{B}$, $\mathbf{A}-2\mathbf{B}$
No numbers are involved.

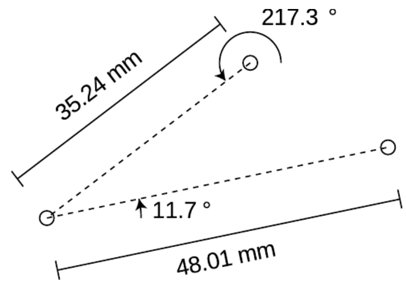


Problem 3-a1.

3-a2 Phnom Penh is 470 km east and 250 km south of Bangkok. Hanoi is 60 km east and 1030 km north of Phnom Penh.
(a) Choose a coordinate system, and translate these data into Δx and Δy values with the proper plus and minus signs.
(b) Find the components of the $\Delta\mathbf{r}$ vector pointing from Bangkok to Hanoi. ✓

3-a3 If you walk 35 km at an angle 25° counterclockwise from east, and then 22 km at 230° counterclockwise from east, find the distance and direction from your starting point to your destination. ✓

3-a4 A machinist is drilling holes in a piece of aluminum according to the plan shown in the figure. She starts with the top hole, then moves to the one on the left, and then to the one on the right. Since this is a high-precision job, she finishes by moving in the direction and at the angle that should take her back to the top hole, and checks that she ends up in the same place. What are the distance and direction from the right-hand hole to the top one? ✓



Problem 3-a4.

3-a5 Suppose someone proposes a new operation in which a vector **A** and a scalar B are added together to make a new vector **C** like this:

$$C_x = A_x + B$$
$$C_y = A_y + B$$
$$C_z = A_z + B$$

Prove that this operation won't be useful in physics, because it's not rotationally invariant.

3-d1 Find the angle between the following two vectors:

$$\hat{\mathbf{x}} + 2\hat{\mathbf{y}} + 3\hat{\mathbf{z}}$$
$$4\hat{\mathbf{x}} + 5\hat{\mathbf{y}} + 6\hat{\mathbf{z}}$$

✓

3-d2 Let a and b be any two numbers (not both zero), and let $\mathbf{u} = a\hat{\mathbf{x}} + b\hat{\mathbf{y}}$. Suppose we want to find a (nonzero) second vector $\mathbf{v}$ in the x-y plane that is perpendicular to $\mathbf{u}$. Use the vector dot product to write down a condition for $\mathbf{v}$ to satisfy, find a suitable $\mathbf{v}$, and check using the dot product that it is indeed a solution.

3-g1 Find a vector that is perpendicular to both of the following two vectors:

$$\hat{\mathbf{x}} + 2\hat{\mathbf{y}} + 3\hat{\mathbf{z}}$$
$$4\hat{\mathbf{x}} + 5\hat{\mathbf{y}} + 6\hat{\mathbf{z}}$$

√

3-g2 Which of the following expressions make sense, and which are nonsense? For those that make sense, indicate whether the result is a vector or a scalar.
(a) $(\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$
(b) $(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}$
(c) $(\mathbf{A} \cdot \mathbf{B}) \times \mathbf{C}$

3-g3 Vector $\mathbf{A} = (3.0\hat{\mathbf{x}} - 4.0\hat{\mathbf{y}})$ meters, and vector $\mathbf{B} = (5.0\hat{\mathbf{x}} + 12.0\hat{\mathbf{y}})$ meters. Find the following: (a) The magnitude of vector $\mathbf{A} - 2\mathbf{B}$. √
(b) The dot product $\mathbf{A} \cdot \mathbf{B}$. √
(c) The cross product $\mathbf{A} \times \mathbf{B}$ (expressing the result in terms of its components). √
(d) The value of $(\mathbf{A} + \mathbf{B}) \cdot (\mathbf{A} - \mathbf{B})$. √
(e) The angle between the two vectors.
√

3-g4 Prove the anticommutative property of the vector cross product, $\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$, using the expressions for the components of the cross product.

3-g5 Label the following statements about vectors as true or false.
(a) The angle between $a\hat{\mathbf{x}} + b\hat{\mathbf{y}}$ and $b\hat{\mathbf{x}} + a\hat{\mathbf{y}}$ is zero.
(b) The three vectors $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{A} + \mathbf{B}$ form a triangle.
(c) The three vectors $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{A} - \mathbf{B}$ form a triangle.
(d) The cross product between two vectors is always perpendicular to each of the two vectors.
(e) If the angle between two vectors is greater than 90°, then the dot product between the two vectors is negative.
(f) A unit vector has magnitude 1 (and no units).

3-g6 Find three vectors with which you can demonstrate that the vector cross product need not be associative, i.e., that $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ need not be the same as $(\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$.

3-g7 Can the vector cross product be generalized to four dimensions? The generalization should, like the three-dimensional version, take two vectors as inputs, give a vector as an output, and be rotationally invariant. (This is of real-world interest because Einstein's theory of relativity can be interpreted as describing time as a kind of fourth dimension.)
⋆

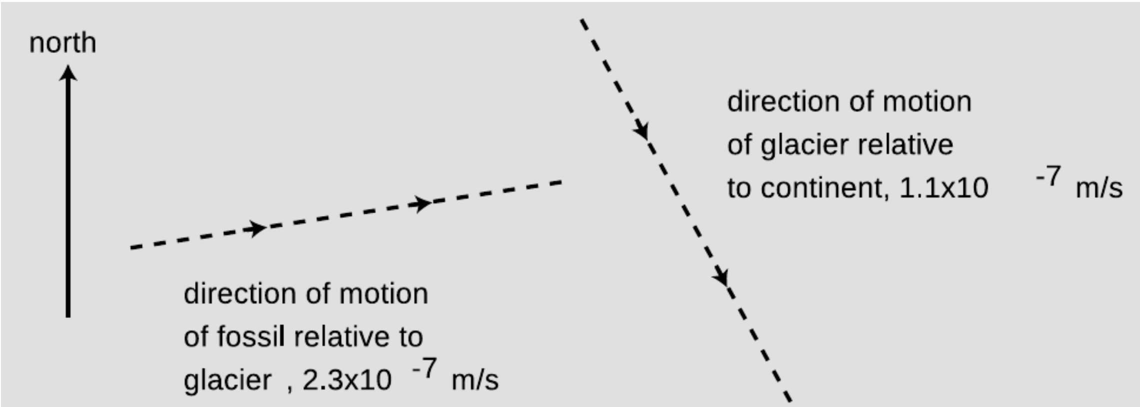
3-g8 A certain function f takes two vectors as inputs and gives an output that is also a vector. The function can be defined in such a way that it is rotationally invariant, and it is also well defined regardless of the units of the vectors. It takes on the following values for the following inputs:

$$f(\hat{\mathbf{x}}, \hat{\mathbf{y}}) = -\hat{\mathbf{z}}$$
$$f(2\hat{\mathbf{x}}, \hat{\mathbf{y}}) = -8\hat{\mathbf{z}}$$
$$f(\hat{\mathbf{x}}, 2\hat{\mathbf{y}}) = -2\hat{\mathbf{z}}$$

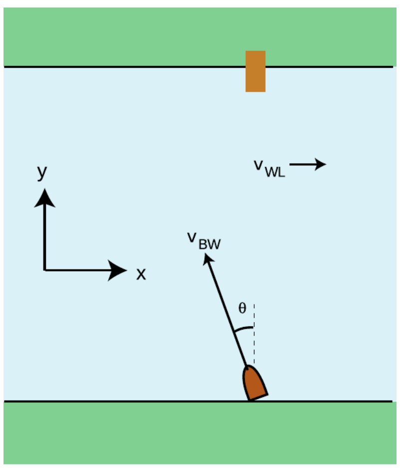
Prove that the given information uniquely determines f, and give an explicit expression for it.
⋆

3-j1 Annie Oakley, riding north on horseback at 30 mi/hr, shoots her rifle, aiming horizontally and to the northeast. The muzzle speed of the rifle is 140 mi/hr. When the bullet hits a defenseless fuzzy animal, what is its speed of impact? Neglect air resistance, and ignore the vertical motion of the bullet.

3-j2 As shown in the figure, you wish to cross a river and arrive at a dock that is directly across from you, but the river's current will tend to carry you downstream. To compensate, you must steer the boat at an angle. Find the angle θ, given the magnitude, $|\mathbf{v}_{WL}|$, of the water's velocity relative to the land, and the maximum speed, $|\mathbf{v}_{BW}|$, of which the boat is capable relative to the water.
▷ Solution, p. 200



Problem 3-j7.



Problem 3-j2.

3-j3 It's a calm day in Los Angeles with no wind. You're in your car on the Ventura Freeway going 105 km/hour when it starts to rain. You notice out the driver's side window that the raindrops make an angle of 70° with respect to the vertical.
(a) What is the speed of the raindrops as measured by someone at rest relative to the freeway? $\checkmark$
(b) What is the speed of the raindrops as measured by you? $\checkmark$

3-j4 A border collie and a rottweiler, both initially at the same location, start chasing two different objects. The border collie starts chasing a stick thrown in the $\hat{\mathbf{x}}+\hat{\mathbf{y}}$ direction, and the rottweiler starts chasing your neighbor in the $-\hat{\mathbf{y}}$ direction. Both dogs move at speed v. For both parts of this problem, give your results as exact expressions, not decimal approximations.
(a) What is the velocity of the rottweiler as measured by the border collie? $\checkmark$
(b) How long does it take for the distance between the two dogs to be D? $\checkmark$

3-j5 A plane can fly at $u = 150$ m/s with respect to the air, and the wind is from the southwest at $v = 50$ m/s.
(a) In what direction should the plane head in order to fly directly north (with respect to the ground)? Give the angle in degrees west of north. $\checkmark$
(b) What is the plane's speed as measured by an observer on the ground? $\checkmark$

3-j6 A plane flies toward a city directly north and a distance D away. The wind speed is u, and the plane's speed with respect to the wind is v.
(a) If the wind is blowing from the west (towards the east), what direction should the plane head

(what angle west of north)? ✓
(b) How long does it take the plane to get to the city? ✓
(c) Check that your answer to part b has units that make sense.
(d) Comment on the behavior of your answer in the case where $u = v$.

3-j7 As shown in the diagram, a dinosaur fossil is slowly moving down the slope of a glacier under the influence of wind, rain and gravity. At the same time, the glacier is moving relative to the continent underneath. The dashed lines represent the directions but not the magnitudes of the velocities. Pick a scale, and use graphical addition of vectors to find the magnitude and the direction of the fossil's velocity relative to the continent. You will need a ruler and protractor. ✓

3-j8 Andrés and Brenda are going to race to see who can first get to a town across a river of width 20.0 m. The water in the river is moving at a constant 0.60 m/s, each person can swim with speed 1.00 m/s with respect to the water, and each person can run 4.00 m/s on land.

Andrés is going to row in such a way that he moves straight towards the town across the river. Brenda, however, decides to get to the other side of the river as quickly as she can, and run.
(a) How long does it take Andrés to swim to the other side of the river? Call this T_A ✓
(b) How long does it take Brenda to get to the other side of the river (not at the town, since the river carries her downstream)? Call this t_1. ✓
(c) How long does it take Brenda to run to the town on the other side of the river? Call this t_2. ✓
(d) How long ($T_B = t_1 + t_2$) does the total trip take Brenda? Who wins the race? ✓

3-j9 César is on one bank of a river in which the water flows at speed w. He can swim at speed u and run at speed v. On the other side, directly across from him, is a town that he wants to reach in the minimum possible time. Depending on the direction in which he chooses to swim, he may need to run some distance along the far bank in order to get to the town. Show that, surprisingly, the optimal angle depends on the variables v and w only through their sum $v + w$. ⋆

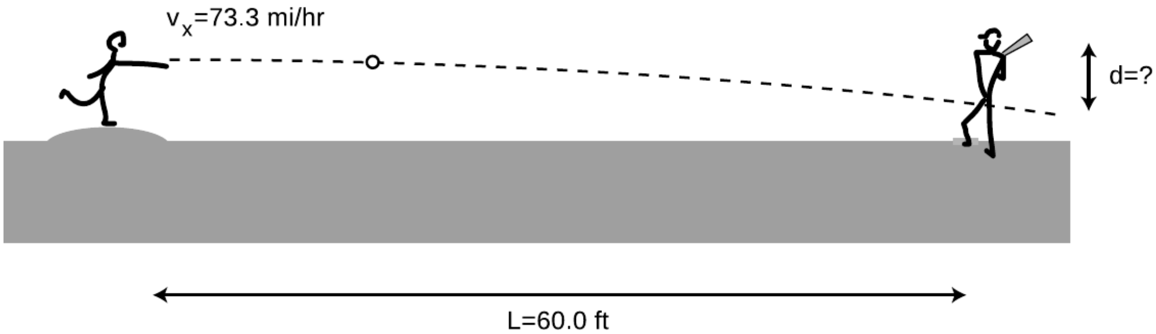
3-m1 Is it possible for a helicopter to have an acceleration due east and a velocity due west? If so, what would be going on? If not, why not?

3-m2 The figure shows the path followed by Hurricane Irene in 2005 as it moved north. The dots show the location of the center of the storm at six-hour intervals, with lighter dots at the time when the storm reached its greatest intensity. Find the time when the storm's center had a velocity vector to the northeast and an acceleration vector to the southeast. Explain.

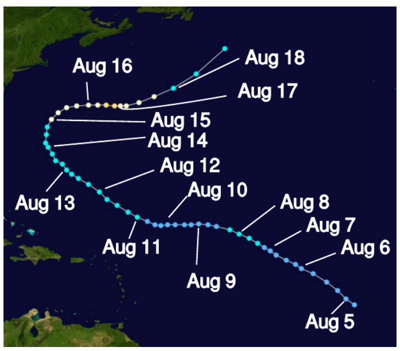
3-m3 A bird is initially flying horizontally east at 21.1 m/s, but one second later it has changed direction so that it is flying horizontally and 7° north of east, at the same speed. What are the magnitude and direction of its acceleration vector during that one second time interval? (Assume its acceleration was roughly constant.) ✓

3-m4 Two cars go over the same speed bump in a parking lot, Maria's Maserati at 25 miles per hour and Park's Porsche at 37. How many times greater is the vertical acceleration of the Porsche? Hint: Remember that acceleration depends both on how much the velocity changes and on how much time it takes to change. ✓

3-p1 Two daredevils, Wendy and Bill, go over Niagara Falls. Wendy sits in an inner tube, and lets the 30 km/hr velocity of the river throw her out horizontally over the falls. Bill paddles a kayak, adding an extra 10 km/hr to his velocity. They go over the edge of the falls at the same moment, side by side. Ignore air friction. Explain your reasoning.
(a) Who hits the bottom first?
(b) What is the horizontal component of Wendy's velocity on impact?
(c) What is the horizontal component of Bill's



Problem 3-s2.



Problem 3-m2.

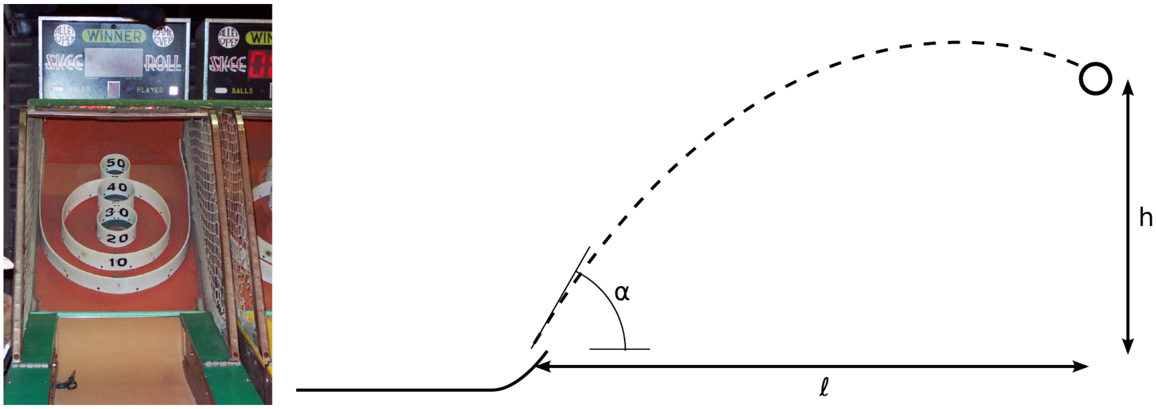
velocity on impact?
(d) Who is going faster on impact?

3-p2 At the 2010 Salinas Lettuce Festival Parade, the Lettuce Queen drops her bouquet while riding on a float moving toward the right. Sketch the shape of its trajectory in her frame of reference, and compare with the shape seen by one of her admirers standing on the sidewalk.

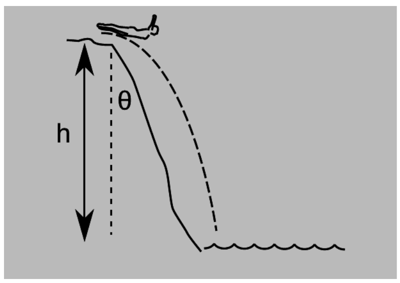
3-s1 A gun is aimed horizontally to the west. The gun is fired, and the bullet leaves the muzzle at $t = 0$. The bullet's position vector as a function of time is $\mathbf{r} = b\hat{\mathbf{x}} + ct\hat{\mathbf{y}} + dt^2\hat{\mathbf{z}}$, where b, c, and d are positive constants.
(a) What units would b, c, and d need to have for the equation to make sense?
(b) Find the bullet's velocity and acceleration as functions of time.
(c) Give physical interpretations of b, c, d, $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$.

3-s2 A baseball pitcher throws a pitch clocked at $v_x = 73.3$ miles/hour. He throws horizontally. By what amount, d, does the ball drop by the time it reaches home plate, $L = 60.0$ feet away?
(a) First find a symbolic answer in terms of L, v_x, and g. ✓
(b) Plug in and find a numerical answer. Express your answer in units of ft. (Note: 1 foot=12 inches, 1 mile=5280 feet, and 1 inch=2.54 cm) ✓

3-s3 You're running off a cliff into a pond. The cliff is $h = 5.0$ m above the water, but the cliff is not strictly vertical; it slopes down to the pond at an angle of $\theta = 20°$ with respect to the vertical. You want to find the minimum speed you need to jump off the cliff in order to land in the water.
(a) Find a symbolic answer in terms of h, θ, and g. ✓
(b) Check that the units of your answer to part a make sense.
(c) Check that the dependence on the variables g, h, and θ makes sense, and check the special cases $\theta = 0$ and $\theta = 90°$.
(d) Plug in numbers to find the numerical result. ✓



Problem 3-s12.



Problem 3-s3.

3-s4 A batter hits a baseball at speed v, at an angle θ above horizontal.
(a) Find an equation for the range (horizontal distance to where the ball falls), R, in terms of the relevant variables. Neglect air friction and the height of the ball above the ground when it is hit.
(b) Interpret your equation in the cases of θ=0 and $\theta = 90°$.
(c) Find the angle that gives the maximum range.

3-s5 A tennis ball is thrown from the ground with speed 15 m/s at an angle of 45° above the horizontal.
(a) How long is the ball in the air (from the throw to when it lands on the ground)? √
(b) What is the maximum height that the ball reaches? √
(c) What is the range of the ball (the horizontal distance the ball has traveled by the time it lands)?
√

3-s6 (a) A ball is thrown straight up with velocity v. Find an equation for the height to which it rises. √
(b) Generalize your equation for a ball thrown at an angle θ above horizontal, in which case its initial velocity components are $v_x = v\cos\theta$ and $v_y = v\sin\theta$. √

3-s7 The first time he played golf, now-deceased North Korean leader Kim Jong-Il is said to have gotten 11 holes in one. Suppose that his son, Kim Jong-Un, wants to top his father's feat by hitting a golf ball all the way from Pyongyang to Seoul, a distance of 195 km. Ignoring air resistance and the curvature of the earth, how fast does he need to hit the ball? Note that the maximum range (assuming no air resistance) is achieved for a launch angle of 45°.

3-s8 Two footballs, one white and one green, are on the ground and kicked by two different footballers. The white ball, which is kicked straight upward with initial speed v_0, rises to height H. The green ball is hit with twice the

initial speed but reaches the same height.
(a) What is the y-component of the green ball's initial velocity vector? Give your answer in terms of v_0 alone. ✓
(b) Which ball is in the air for a longer amount of time?
(c) What is the range of the green ball? Your answer should only depend on H.
✓

3-s9 You throw a rock horizontally from the edge of the roof of a building of height 10.0 m. The rock hits the ground at exactly twice its initial speed. How fast was the rock thrown off the roof? Express your answer to three significant figures.
✓

3-s10 You throw a rock horizontally from the edge of the roof of a building of height h with speed v_0. What is the (positive) angle between the final velocity vector and the horizontal when the rock hits the ground?
✓

3-s11 Standing on the edge of the roof of a building of height h, you throw a rock with speed v_0 at 30° above the horizontal.
(a) How high above the ground does the rock get? ✓
(b) How far away from the building does the rock land?
✓

3-s12 The figure shows an arcade game called skee ball that is similar to bowling. The player rolls the ball down a horizontal alley. The ball then rides up a curved lip and is launched at an initial speed u, at an angle α above horizontal. Suppose we want the ball to go into a hole that is at horizontal distance ℓ and height h, as shown in the figure.
(a) Find the initial speed u that is required, in terms of the other variables and g. ✓
(b) Check that your answer to part a has units that make sense.
(c) Check that your answer to part a depends on g in a way that makes sense. This means that you should first determine on physical grounds whether increasing g should increase u, or decrease it. Then see whether your answer to part a has this mathematical behavior.
(d) Do the same for the dependence on h.
(e) Interpret your equation in the case where $\alpha = 90°$.
(f) Interpret your equation in the case where $\tan\alpha = h/\ell$.
(g) Find u numerically if $h = 70$ cm, $\ell = 60$ cm, and $\alpha = 65°$. ✓

3-s13 A particle leaves point P at time $t = 0$ s with initial velocity $(-2.0\hat{\mathbf{x}} + 4.0\hat{\mathbf{y}})$ m/s. Point P is located on the x axis at position $(x, y) = (10.0\text{ m}, 0)$. If the particle experiences constant acceleration $\mathbf{a} = (-5.0\hat{\mathbf{y}})$ m/s^2, then which axis does it cross first, x or y, and at what location?

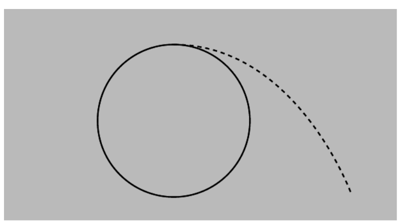
3-s14 A Hot Wheels car is rolling along a horizontal track at speed $v_0 = 6.0$ m/s. It then comes to a ramp inclined at an angle $\theta = 30°$ above the horizontal, and the car undergoes a deceleration of $g\sin\theta = 4.9$ m/s^2 when moving along the ramp. The track ends at the top of the ramp, so the car is launched into the air. By the time the car reaches the top of the ramp, its speed has gone down to 3.0 m/s.
(a) How high is the top of the ramp (vertical height, not distance along the ramp)? ✓
(b) After the car is achieves lift-off, how long does it spend in the air before hitting the ground?
✓

3-s15 A Hot Wheels car is rolling along a horizontal track at speed v_0. It then comes to a ramp inclined at an angle θ above the horizontal. The car undergoes a deceleration of $g\sin\theta$ while rolling up the ramp. The track ends after a distance L, so the car is launched into the air.
(a) What is the speed of the car when it leaves the ramp? ✓
(b) How high does the car get above the ground?
✓

3-s16 The figure shows a vertical cross-section of a cylinder. A gun at the top shoots a bullet horizontally. What is the minimum speed

at which the bullet must be shot in order to completely clear the cylinder?

⋆



Problem 3-s16.

4 Newton's laws, part 1

This is not a textbook. It's a book of problems meant to be used along with a textbook. Although each chapter of this book starts with a brief summary of the relevant physics, that summary is not meant to be enough to allow the reader to actually learn the subject from scratch. The purpose of the summary is to show what material is needed in order to do the problems, and to show what terminology and notation are being used.

4.1 Newton's first law

Isaac Newton (1643-1727) originated the idea of explaining all events, both on earth and in the heavens, using a set of simple and universal mathematical laws. His three laws talk about *forces*, so what is a force?

In figure 4.1, the legendary Baron von Munchausen lifts himself and his horse out of a swamp by pulling up on his own pigtail. This is not actually possible, because an object can't accelerate by exerting a force on itself. A force is always an interaction between *two* objects.

Figure 4.1: Escaping from a swamp.

The left side of figure 4.2 shows a hand making a force on a rope. Two objects: hand and rope.

A force refers to a direct cause, not an indirect one. A pool player makes a force on the cue stick, but not on the cue ball.

To finish defining what we mean by a force, we need to say how we would measure a force numerically. In the right-hand side, the stretching of a spring is a measure of the hand's force. The SI unit of force is the newton (N), which we will see later is actually defined in a convenient way in terms of the base units of the SI. Force is a vector.

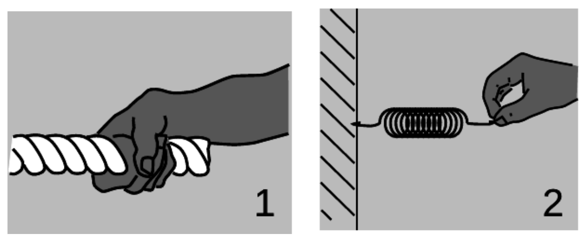


Figure 4.2: Forces.

Suppose that we can prevent any forces at all from acting on an object, perhaps by moving it far away from all other objects, or surrounding it with shielding. (For example, there is a nickel-iron alloy marketed as "mu-metal" which blocks magnetic forces very effectively.) *Newton's first law* states that in this situation, the object has a zero acceleration vector, i.e., its velocity vector is constant. If the object is already at rest, it remains at rest. If it is already in motion, it remains in motion at constant speed in the same direction.

Newton's first law is a more detailed and quantitative statement of the law of inertia. The first law holds in an inertial frame of reference; in fact, this is just a restatement of what we mean by an inertial frame.

The first law may not seem very useful for applications near the earth's surface, since an object there will always be subject at least to the

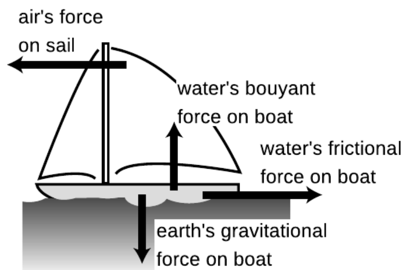


Figure 4.3: The four forces on the sailboat cancel out.

force of gravity. But the first law can also be extended to apply to cases in which forces do act on an object, but they cancel out. An example is the sailboat in figure 4.3.

An object can rotate or change its shape. A cat does both of these things when it falls and brings its feet under itself before it hits the ground. In such a situation, it is not immediately obvious what is meant by "the" velocity of the object. We will see later that Newton's first law can still be made to hold in such cases if we measure its motion by using a special point called its center of mass, which is the point on which it would balance. In the example of Baron von Munchausen, it is certainly possible for one part of his body to accelerate another part of his body by making a force on it; however, this will have no effect on the motion of his center of mass.

4.2 Newton's second law

Newton's second law tells us what happens when the forces acting on an object do *not* cancel out. The object's acceleration is then given by

$$\mathbf{a} = \frac{\mathbf{F}_{\text{total}}}{m}, \tag{4.1}$$

where F_{total} is the vector sum of all the forces, and m is the object's mass. Mass is a permanent property of an object that measures its inertia, i.e., how much it resists a change in its motion. Since the SI unit of mass is the kilogram, it follows from Newton's second law that the newton is related to the base units of the SI as $1\ \text{N} = 1\ \text{kg}\cdot\text{m/s}^2$.

The force that the earth's gravity exerts on an object is called its *weight*, which is not the same thing as its mass. An object of mass m has weight mg (problem 4-a1, p. 46).

4.3 Newton's third law

We have seen that a force is always an interaction between two objects. *Newton's third law* states that these forces come in pairs. If object A exerts a force on object B, then B also exerts a force on A. The two forces have equal magnitudes but are in opposite directions. In symbols,

$$\mathbf{F}_{\text{A on B}} = -\mathbf{F}_{\text{B on A}}. \tag{4.2}$$

Newton's third law holds regardless of whether everything is in a state of equilibrium. It might seem as though the two forces would cancel out, but they can't cancel out because it doesn't even make sense to add them in the first place. They act on different objects, and it only makes sense to add forces if they act on the same object.

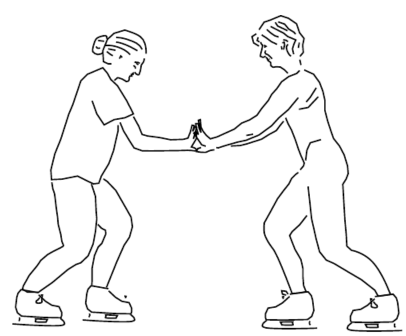


Figure 4.4: Newton's third law does not mean that forces always cancel out so that nothing can ever move. If these two ice skaters, initially at rest, push against each other, they will both move.

The pair of forces related by Newton's third law are always of the same type. For example, the hand in the left side of figure 4.2 makes a frictional force to the right on the rope. Newton's third law tells us that the rope exerts a force on the hand that is to the left and of the same strength. Since one of these forces is frictional, the other is as well.

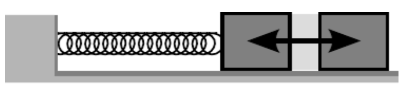
Problems

4-a1 (a) Why would it not make sense to say that the force of gravity acting on an object equals g?
(b) Why would it not make sense to say that the force of gravity acting on an object of mass m equals m?
(c) Use Newton's second law to prove that the magnitude of the gravitational force acting on an object of mass m equals mg.

▷ Solution, p. 200

4-a2 You are given a large sealed box, and are not allowed to open it. Which of the following experiments measure its mass, and which measure its weight? [Hint: Which experiments would give different results on the moon?]
(a) Put it on a frozen lake, throw a rock at it, and see how fast it scoots away after being hit.
(b) Drop it from a third-floor balcony, and measure how loud the sound is when it hits the ground.
(c) As shown in the figure, connect it with a spring to the wall, and watch it vibrate.

▷ Solution, p. 201



Problem 4-a2.

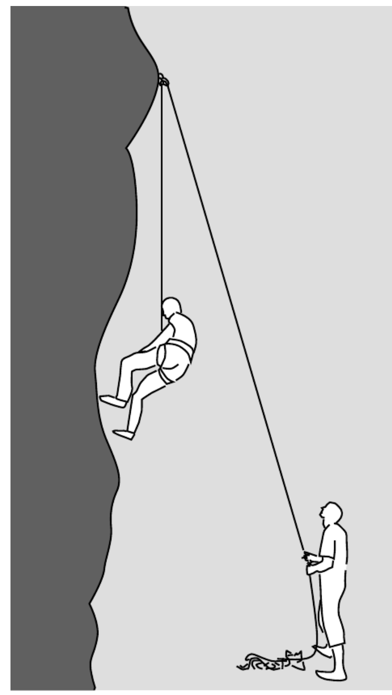
4-a3 (a) Compare the mass of a one-liter water bottle on earth, on the moon, and in interstellar space.
(b) Do the same for its weight.

4-a4 In the figure, the rock climber has finished the climb, and his partner is lowering him back down to the ground at approximately constant speed. The following is a student's analysis of the forces acting *on the climber*. The arrows give the directions of the forces.

force of the earth's gravity, $\downarrow$
force from the partner's hands, $\uparrow$
force from the rope, $\uparrow$

The student says that since the climber is moving down, the sum of the two upward forces must be slightly less than the downward force of gravity.

Correct all mistakes in the above analysis.



Problem 4-a4.

4-a5 A car is accelerating forward along a straight road. If the force of the road on the car's wheels, pushing it forward, is a constant 3.0 kN, and the car's mass is 1000 kg, then how long will the car take to go from 20 m/s to 50 m/s?

▷ Solution, p. 201

4-a6 An object is observed to be moving at constant speed in a certain direction. Can you conclude that no forces are acting on it? Explain. [Based on a problem by Serway and Faughn.]

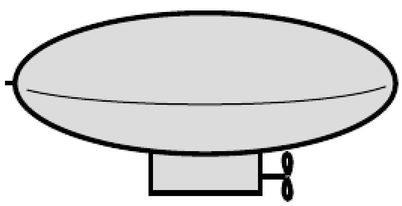
4-a7 A book is pushed along a frictionless table with a constant horizontal force. The book starts from rest and travels 2.0 m in 1.0 s. If

the same force continues, how far will the book travel in the next 1.0 s?

√

4-d1 A blimp is initially at rest, hovering, when at $t = 0$ the pilot turns on the engine driving the propeller. The engine cannot instantly get the propeller going, but the propeller speeds up steadily. The steadily increasing force between the air and the propeller is given by the equation $F = kt$, where k is a constant. If the mass of the blimp is m, find its position as a function of time. (Assume that during the period of time you're dealing with, the blimp is not yet moving fast enough to cause a significant backward force due to air resistance.)

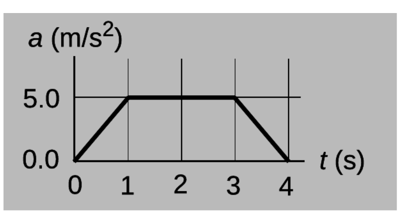
√



Problem 4-d1.

4-g1 The acceleration of a 1.0 kg object is given in the graph.
(a) What is the *maximum* force that acts on the object over the time interval $[0.0, 4.0]$ s? √
(b) What is the *average* force over the same interval?

√



Problem 4-g1.

4-g2 Flaca has brought a bathroom scale with her in an elevator and is standing on it. Just before the elevator arrives at the top floor, as the car is slowing down, she notices that according to the scale, her weight appears to be off by 3% from its normal value W.
(a) Does the scale read $0.97W$, or $1.03W$? √
(b) What is the magnitude of the acceleration of the elevator?

√

4-g3 A person who normally weighs 890 N is standing on a scale inside an elevator. The elevator is moving upward with a speed of 10 m/s, and then begins to decelerate at a rate of 5.0 m/s^2.
(a) Before the elevator begins to decelerate, what is the reading on the scale? √
(b) What about while the elevator is slowing down?

√

4-g4 A bullet of mass m is fired from a pistol, accelerating from rest to a speed v in the barrel's length L.
(a) What is the force on the bullet? (Assume this force is constant.) √
(b) Check that the units of your answer to part a make sense.
(c) Check that the dependence of your answer on each of the three variables makes sense.

4-g5 You push a cup of mass M across a table, using a force of magnitude F. Because of a second, frictional force, the cup's acceleration only has magnitude $F/3M$. What is the magnitude of this frictional force?

√

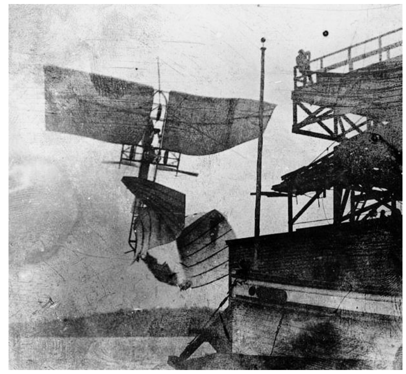
4-g6 In an experiment, a force is applied to two different unknown masses. This force causes the first object, with mass m_1, to have acceleration a_1, and gives an object of mass m_2 an acceleration a_2, where $a_1 > a_2$.
(a) Which mass is heavier: m_1 or m_2?
(b) Based on the experimental data (a_1 and a_2), what acceleration would the force give to an object of mass $m_1 + m_2$?

√

4-j1 At low speeds, every car's acceleration is limited by traction, not by the engine's power. Suppose that at low speeds, a certain car is normally capable of an acceleration of 3 m/s^2. If

it is towing a trailer with half as much mass as the car itself, what acceleration can it achieve? [Based on a problem from PSSC Physics.]

4-j2 The tires used in Formula 1 race cars can generate traction (i.e., force from the road) that is as much as 1.9 times greater than with the tires typically used in a passenger car. Suppose that we're trying to see how fast a car can cover a fixed distance starting from rest, and traction is the limiting factor. By what factor is this time reduced when switching from ordinary tires to Formula 1 tires?
√



Problem 4-j3.

4-j3 At the turn of the 20th century, Samuel Langley engaged in a bitter rivalry with the Wright brothers to develop human flight. Langley's design used a catapult for launching. For safety, the catapult was built on the roof of a houseboat, so that any crash would be into the water. This design required reaching cruising speed within a fixed, short distance, so large accelerations were required, and the forces frequently damaged the craft, causing dangerous and embarrassing accidents. Langley achieved several uncrewed, unguided flights, but never succeeded with a human pilot. If the force of the catapult is fixed by the structural strength of the plane, and the distance for acceleration by the size of the houseboat, by what factor is the launch velocity reduced when the plane's 340 kg is augmented by the 60 kg mass of a small man?
√

4-j4 In the 1964 Olympics in Tokyo, the best men's high jump was 2.18 m. Four years later in Mexico City, the gold medal in the same event was for a jump of 2.24 m. Because of Mexico City's altitude (2400 m), the acceleration of gravity there is lower than that in Tokyo by about 0.01 m/s^2. Suppose a high-jumper has a mass of 72 kg.
(a) Compare his mass and weight in the two locations.
(b) Assume that he is able to jump with the same initial vertical velocity in both locations, and that all other conditions are the same except for gravity. How much higher should he be able to jump in Mexico City? √
(Actually, the reason for the big change between '64 and '68 was the introduction of the "Fosbury flop.")

4-j5 Your friend, who's kind of an idiot, jumps out of a third-story window. After falling 7.0 m, he lands on his stomach so that as his body compresses on impact, his center of mass only moves 0.020 m. What is the average force of the ground on your friend as he smacks the floor? Express your answer in terms of his weight W.
√

4-j6 A book is pushed along a frictionless table with a constant horizontal force. The book starts from rest and travels 2.0 m in 1.0 s. If the same experiment is carried out again, but with a book having twice the mass, how far will this heavier book travel?
√

4-m1 In each case, identify the force that causes the acceleration, and give its Newton's-third-law partner. Describe the effect of the partner force. (a) A swimmer speeds up. (b) A golfer hits the ball off of the tee. (c) An archer fires an arrow. (d) A locomotive slows down.
▷ Solution, p. 201

4-m2 A little old lady and a pro football player collide head-on. Compare their forces on

each other, and compare their accelerations. Explain.

Problem 4-m2.

4-m3 The earth is attracted to an object with a force equal and opposite to the force of the earth on the object. If this is true, why is it that when you drop an object, the earth does not have an acceleration equal and opposite to that of the object?

4-m4 When you stand still, there are two forces acting on you, the force of gravity (your weight) and the normal force of the floor pushing up on your feet. Are these forces equal and opposite? Does Newton's third law relate them to each other? Explain.

4-m5 Some garden shears are like a pair of scissors: one sharp blade slices past another. In the "anvil" type, however, a sharp blade presses against a flat one rather than going past it. A gardening book says that for people who are not very physically strong, the anvil type can make it easier to cut tough branches, because it concentrates the force on one side. Evaluate this claim based on Newton's laws. [Hint: Consider the forces acting on the branch, and the motion of the branch.]

$\star$

4-m6 Pick up a heavy object such as a backpack or a chair, and stand on a bathroom scale. Shake the object up and down. What do you observe? Interpret your observations in terms of Newton's third law.

$\star$

4-p1 (a) Let T be the maximum tension that an elevator's cable can withstand without breaking, i.e., the maximum force it can exert. If the motor is programmed to give the car an acceleration a ($a > 0$ is upward), what is the maximum mass that the car can have, including passengers, if the cable is not to break? $\checkmark$
(b) Interpret the equation you derived in the special cases of $a = 0$ and of a downward acceleration of magnitude g.

4-p2 While escaping from the palace of the evil Martian emperor, Sally Spacehound jumps from a tower of height h down to the ground. Ordinarily the fall would be fatal, but she fires her blaster rifle straight down, producing an upward force of magnitude F_B. This force is insufficient to levitate her, but it does cancel out some of the force of gravity. During the time t that she is falling, Sally is unfortunately exposed to fire from the emperor's minions, and can't dodge their shots. Let m be her mass, and g the strength of gravity on Mars.
(a) Find the time t in terms of the other variables.
(b) Check the units of your answer to part a.
(c) For sufficiently large values of F_B, your answer to part a becomes nonsense — explain what's going on. $\checkmark$

4-p3 A helicopter of mass m is taking off vertically. The only forces acting on it are the earth's gravitational force and the force, F_{air}, of the air pushing up on the propeller blades.
(a) If the helicopter lifts off at $t = 0$, what is its vertical speed at time t?
(b) Check that the units of your answer to part a make sense.
(c) Discuss how your answer to part a depends on all three variables, and show that it makes sense. That is, for each variable, discuss what

would happen to the result if you changed it while keeping the other two variables constant. Would a bigger value give a smaller result, or a bigger result? Once you've figured out this *mathematical* relationship, show that it makes sense *physically*.
(d) Plug numbers into your equation from part a, using $m = 2300$ kg, $F_{air} = 27000$ N, and $t = 4.0$ s.

✓

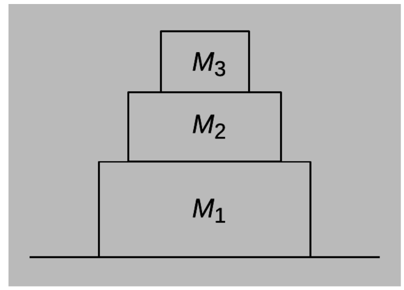
4-p4 A uranium atom deep in the earth spits out an alpha particle. An alpha particle is a fragment of an atom. This alpha particle has initial speed v, and travels a distance d before stopping in the earth.
(a) Find the force, F, from the dirt that stopped the particle, in terms of v, d, and its mass, m. Don't plug in any numbers yet. Assume that the force was constant. ✓
(b) Show that your answer has the right units.
(c) Discuss how your answer to part a depends on all three variables, and show that it makes sense. That is, for each variable, discuss what would happen to the result if you changed it while keeping the other two variables constant. Would a bigger value give a smaller result, or a bigger result? Once you've figured out this *mathematical* relationship, show that it makes sense *physically*.
(d) Evaluate your result for $m = 6.7 \times 10^{-27}$ kg, $v = 2.0 \times 10^4$ km/s, and $d = 0.71$ mm. ✓

4-p5 A car is pushing a truck from behind. The car has mass M, and the truck has mass $3M$. If the maximum force that the ground can provide to the cars' tires is F, what is the maximum force between the two vehicles? Assume no other horizontal forces act on the truck.

✓

4-p6 Blocks of mass M_1, M_2, and M_3 are stacked on a table as shown in the figure. Let the upward direction be positive.
(a) What is the force on block 2 from block 3? ✓
(b) What is the force on block 2 from block 1?

✓



Problem 4-p6.

4-s1 When I cook rice, some of the dry grains always stick to the measuring cup. To get them out, I turn the measuring cup upside-down and hit the "roof" with my hand so that the grains come off of the "ceiling." (a) Explain why static friction is irrelevant here. (b) Explain why gravity is negligible. (c) Explain why hitting the cup works, and why its success depends on hitting the cup hard enough.

⋆

4-s2 The following reasoning leads to an apparent paradox; explain what's wrong with the logic. A baseball player hits a ball. The ball and the bat spend a fraction of a second in contact. During that time they're moving together, so their accelerations must be equal. Newton's third law says that their forces on each other are also equal. But $a = F/m$, so how can this be, since their masses are unequal? (Note that the paradox isn't resolved by considering the force of the batter's hands on the bat. Not only is this force very small compared to the ball-bat force, but the batter could have just thrown the bat at the ball.)

⋆

5 Newton's laws, part 2

This is not a textbook. It's a book of problems meant to be used along with a textbook. Although each chapter of this book starts with a brief summary of the relevant physics, that summary is not meant to be enough to allow the reader to actually learn the subject from scratch. The purpose of the summary is to show what material is needed in order to do the problems, and to show what terminology and notation are being used.

5.1 Classification of forces

A fundamental and still unsolved problem in physics is the classification of the forces of nature. Ordinary experience suggests to us that forces come in different types, which behave differently. Frictional forces seem clearly different from magnetic forces.

But some forces that appear distinct are actually the same. For instance, the friction that holds a nail into the wall seems different from the kind of friction that we observe when fluids are involved — you can't drive a nail into a waterfall and make it stick. But at the atomic level, both of these types of friction arise from atoms bumping into each other. The force that holds a magnet on your fridge also seems different from the force that makes your socks cling together when they come out of the dryer, but it was gradually realized, starting around 1800 and culminating with Einstein's theory of relativity in 1905, that electricity and magnetism are actually closely related things, and observers in different states of motion do not even agree on what is an electric force and what is a magnetic one. The tendency has been for more and more superficially disparate forces to become unified in this way.

Today we have whittled the list down to only three types of interactions at the subatomic level (called the gravitational, electroweak, and strong forces). It is possible that some future theory of physics will reduce the list to only one — which Star Wars fans would then probably want to call "The Force."

Nevertheless, there is a practical classification of forces that works pretty well for objects on the human scale, and that is usually more convenient. Figure **??** on p. **??** shows this scheme in the form of a tree.

5.2 Friction

If you push a refrigerator across a kitchen floor, you will find that as you make more and more force, at first the fridge doesn't move, but that eventually when you push hard enough, it unsticks and starts to slide. At the moment of unsticking, static friction turns into kinetic friction. Experiments support the following approximate model of friction when the objects are solid, dry, and rigid. We have two unitless coefficients μ_s and μ_k, which depend only on the types of surfaces. The maximum force of static friction is limited to

$$F_s \leq \mu_s F_N, \tag{5.1}$$

where F_N is the normal force between the surfaces, i.e., the amount of force with which they are being pressed together. Kinetic friction is given by

$$F_k = \mu_k F_N. \tag{5.2}$$

5.3 Elasticity

When a force is applied to a solid object, it will change its shape, undergoing some type of deformation such as flexing, compression, or expansion. If the force is small enough, then this change is proportional to the force, and when the force is removed the object will resume its original shape. A simple example is a spring being stretched or compressed. If the spring's relaxed length is x_0, then its length x is related to the

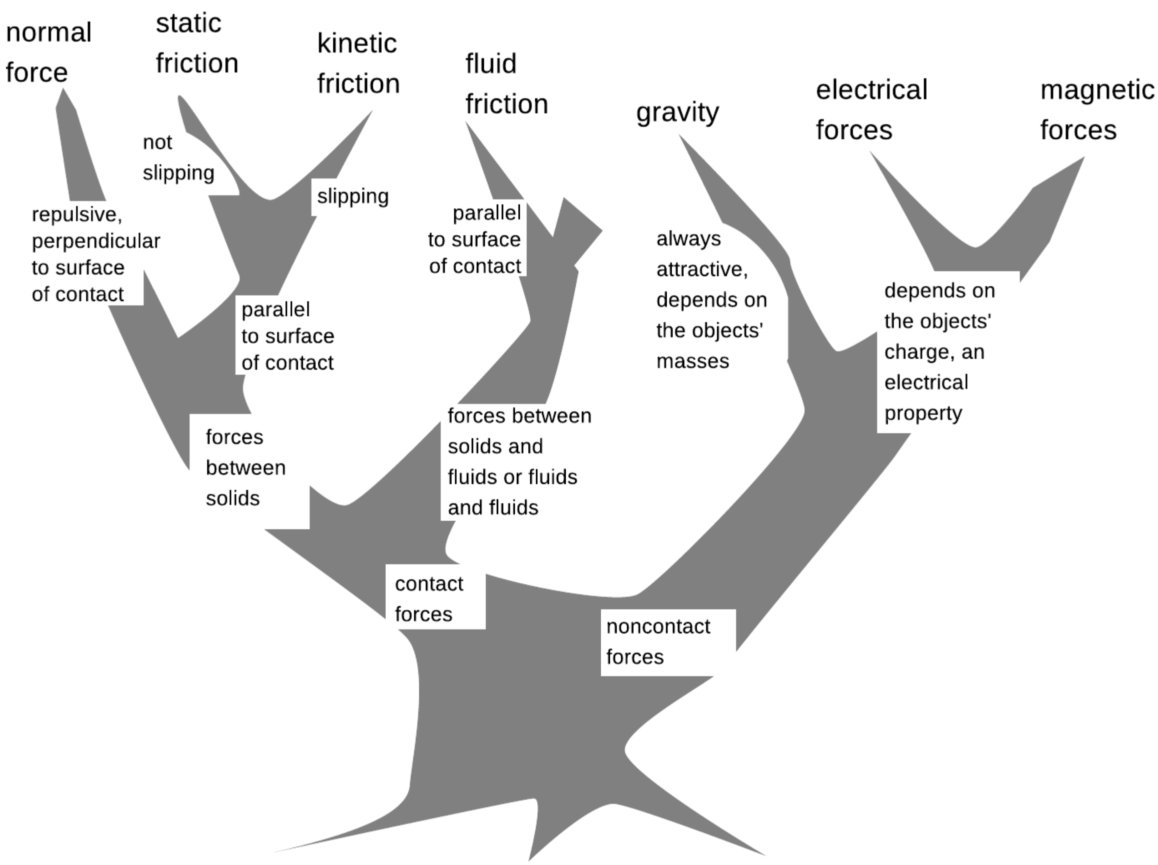


Figure 5.1: A practical classification scheme for forces.

force applied to it by *Hooke's law*,

$$F \approx k(x - x_0). \tag{5.3}$$

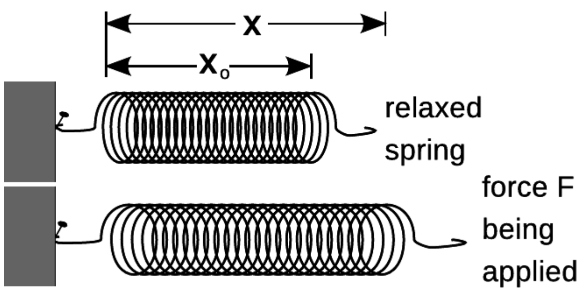


Figure 5.2: Hooke's law.

5.4 Ropes, pulleys, tension, and simple machines

If you look carefully at a piece of rope or yarn while tightening it, you will see a physical change in the fibers. This is a manifestation of the fact that there is tension in the rope. Tension is a scalar with units of newtons. For a rope of negligible mass, the tension is constant throughout the rope, and it equals the magnitudes of the forces at its ends. This is still true if the rope goes around a frictionless post, or a massless pulley with a frictionless axle. You can't push with a rope, you can only pull. A rigid object such as a pencil can, however, sustain compression, which is equivalent to negative tension.

A pulley is an example of a *simple machine*, which is a device that can amplify a force by some factor, while reducing the amount of motion by the inverse of that factor. Another example of a simple machine is the gear system on a bicycle.

We can put more than one simple machine together in order to give greater amplification of forces or to redirect forces in different directions. For an idealized system,[1] the fundamental principles are:

1. The total force acting on any pulley is zero.[2]
2. The tension in any given piece of rope is constant throughout its length.
3. The length of every piece of rope remains the same.

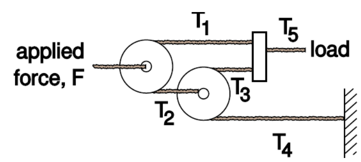


Figure 5.3: A complicated pulley system. The bar is massless.

As an example, let us find the mechanical advantage T_5/F of the pulley system shown in figure 5.3. By rule 2, $T_1 = T_2$, and by rule 1, $F = T_1 + T_2$, so $T_1 = T_2 = F/2$. Similarly, $T_3 = T_4 = F/4$. Since the bar is massless, the same reasoning that led to rule 1 applies to the bar as well, and $T_5 = T_1 + T_3$. The mechanical advantage is $T_5/F = 3/4$, i.e., this pulley system *reduces* the input force.

5.5 Analysis of forces

Newton's second law refers to the total force acting on a particular object. Therefore whenever we want to apply the second law, a necessary preliminary step is to pick an object and list all the forces acting on it. In addition, it may be helpful to determine the types and directions of the forces and also to identify the Newton's-third-law partners of those forces, i.e., all the forces that our object exerts back on other things.

[1] In such a system: (1) The ropes and pulleys have negligible mass. (2) Friction in the pulleys' bearings is negligible. (3) The ropes don't stretch.

[2] $F = ma$, and $m = 0$ since the pulley's mass is assumed to be negligible.

force acting on Fifi	*force related to it by Newton's third law*
planet earth's gravitational force $F_W = mg$ on Fifi, $\downarrow$	Fifi's gravitational force on earth, $\uparrow$
belt's kinetic frictional force F_k on Fifi, $\rightarrow$	Fifi's kinetic frictional force on belt, $\leftarrow$
belt's normal force F_N on Fifi, $\uparrow$	Fifi's normal force on belt, $\downarrow$

Table 5.1: Analysis of the forces on the dog.

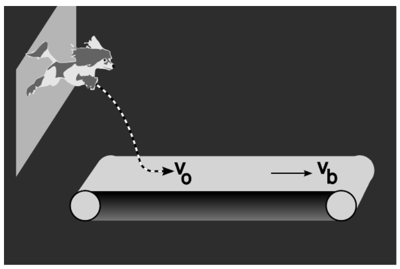


Figure 5.4: The spy dog lands on the moving conveyor belt.

As an example, consider figure 5.4. Fifi is an industrial espionage dog who loves doing her job and looks great doing it. She leaps through a window and lands at initial horizontal speed v_o on a conveyor belt which is itself moving at the greater speed v_b. Unfortunately the coefficient of kinetic friction μ_k between her foot-pads and the belt is fairly low, so she skids, and the effect on her coiffure is *un désastre*. Table 5.1 shows the resulting analysis of the forces in which she participates.

Problems

In problems 5-a1-5-a5, analyze the forces using a table in the format shown in section 5.5 on p. 53. Analyze the forces in which the italicized object participates.

5-a1 Some people put a spare car key in a little magnetic *box* that they stick under the chassis of their car. Let's say that the box is stuck directly underneath a horizontal surface, and the car is parked. (See instructions above.)

5-a2 Analyze two examples of *objects* at rest relative to the earth that are being kept from falling by forces other than the normal force. Do not use objects in outer space, and do not duplicate problem 5-a1 or 5-a5. (See instructions above.)

5-a3 A *person* is rowing a boat, with her feet braced. She is doing the part of the stroke that propels the boat, with the ends of the oars in the water (not the part where the oars are out of the water). (See instructions above.)

5-a4 A *farmer* is in a stall with a cow when the cow decides to press him against the wall, pinning him with his feet off the ground. Analyze the forces in which the farmer participates. (See instructions above.)

5-a5 A propeller *plane* is cruising east at constant speed and altitude. (See instructions above.)

5-a6 Someone tells you she knows of a certain type of Central American earthworm whose skin, when rubbed on polished diamond, has $\mu_k > \mu_s$. Why is this not just empirically unlikely but logically suspect?

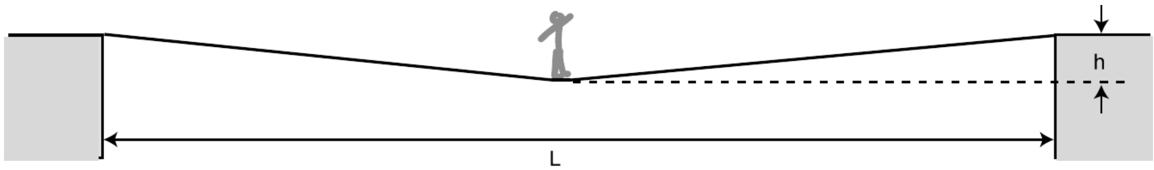
⋆

5-d1 The figure shows a boy hanging in three positions: (1) with his arms straight up, (2) with his arms at 45 degrees, and (3) with his arms at 60 degrees with respect to the vertical. Compare the tension in his arms in the three cases.

5-d2 For safety, mountain climbers often wear a climbing harness and tie in to other climbers on a rope team or to anchors such as pitons or snow anchors. When using anchors, the climber usually wants to tie in to more than one, both for extra strength and for redundancy in case one fails. The figure shows such an arrangement, with the climber hanging from a pair of anchors forming a "Y" at an angle θ. The metal piece at the center is called a carabiner. The usual advice is to make $\theta < 90°$; for large values of θ, the stress placed on the anchors can be many times greater than the actual load L, so that two anchors are actually *less* safe than one.
(a) Find the force S at each anchor in terms of L and θ. ✓
(b) Verify that your answer makes sense in the case of $\theta = 0$.
(c) Interpret your answer in the case of $\theta = 180°$.
(d) What is the smallest value of θ for which S equals or exceeds L, so that for larger angles a failure of at least one anchor is *more* likely than it would have been with a single anchor? ✓

5-d3 Problem 5-d2 discussed a possible correct way of setting up a redundant anchor for mountaineering. The figure for this problem shows an incorrect way of doing it, by arranging the rope in a triangle (which we'll take to be isoceles). One of the bad things about the triangular arrangement is that it requires more force from the anchors, making them more likely to fail. (a) Using the same notation as in problem 5-d2, find S in terms of L and θ. ✓
(b) Verify that your answer makes sense in the case of $\theta = 0$, and compare with the correct setup.

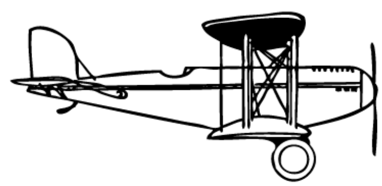
5-d4 A person of mass M stands in the middle of a tightrope, which is fixed at the ends to two buildings separated by a horizontal distance L. The rope sags in the middle, stretching and



Problem 5-d4.

Problem 5-a3.

Problem 5-a4.

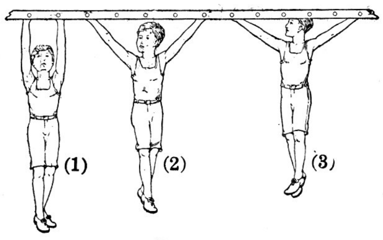


Problem 5-a5.

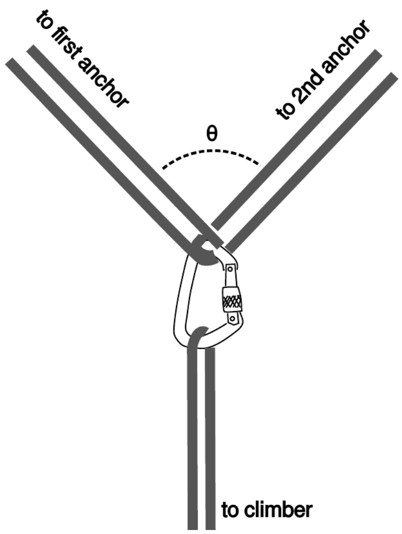
lengthening the rope slightly.
(a) If the tightrope walker wants the rope to sag vertically by no more than a height h, find the minimum tension, T, that the rope must be able to withstand without breaking, in terms of h, g, M, and L. ✓
(b) Based on your equation, explain why it is not possible to get $h = 0$, and give a physical interpretation.

5-d5 The angle of repose is the maximum slope on which an object will not slide. On airless, geologically inert bodies like the moon or an asteroid, the only thing that determines whether dust or rubble will stay on a slope is whether the slope is less steep than the angle of repose.
(a) Find an equation for the angle of repose, deciding for yourself what are the relevant variables.
(b) On an asteroid, where g can be thousands of times lower than on Earth, would rubble be able to lie at a steeper angle of repose?

5-d6 Your hand presses a block of mass m against a wall with a force $\mathbf{F}_H$ acting at an angle θ, as shown in the figure. Find the minimum and maximum possible values of $|\mathbf{F}_H|$ that can keep the block stationary, in terms of m, g, θ, and μ_s, the coefficient of static friction between the block and the wall. Check both your answers in



Problem 5-d1.



Problem 5-d2.

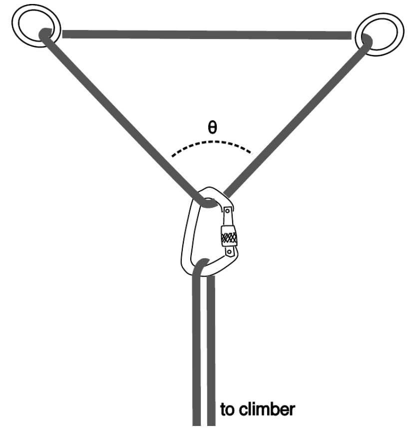
the case of $\theta = 90°$, and interpret the case where the maximum force is infinite.

√ ⋆

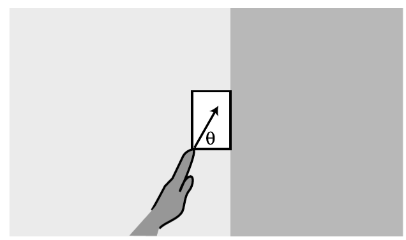
5-d7 A telephone wire of mass m is strung between two poles, making an angle θ with the horizontal at each end. (a) Find the tension at the center. √
(b) Which is greater, the tension at the center or at the ends?

⋆

5-d8 The photo shows a coil of rope wound around a smooth metal post. A large amount of tension is applied at the bottom of the coil, but only a tiny force, supplied by a piece of sticky tape, is needed at the top to keep the rope from slipping. Show that the ratio of these two forces



Problem 5-d3.

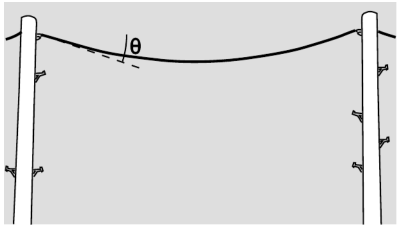


Problem 5-d6.

increases exponentially with the number of turns of rope, and find an expression for that ratio.

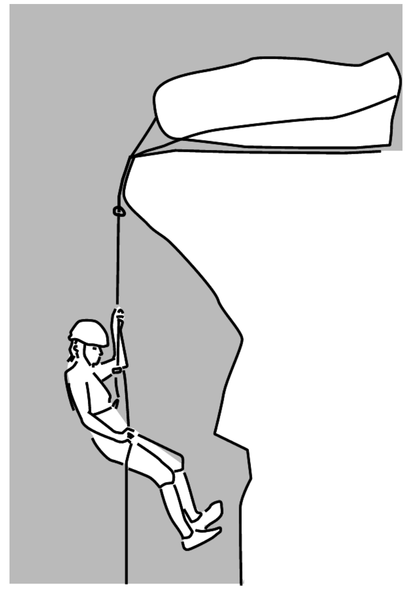
√ ⋆⋆

5-d9 The figure shows a mountaineer doing a vertical rappel. Her anchor is a big boulder. The American Mountain Guides Association suggests as a rule of thumb that in this situation, the boulder should be at least as big as a refrigerator, and should be sitting on a surface that is horizontal rather than sloping. The goal of this problem is to estimate what coefficient of static friction μ_s between the boulder and the ledge is required if this setup is to hold the person's body weight. For comparison, reference books meant for civil engineers building walls out of granite blocks state that granite on granite typically has a $\mu_s \approx 0.6$. We expect the result of our calculation to be much less than this, both because a large margin of safety



Problem 5-d7.

Problem 5-d8.



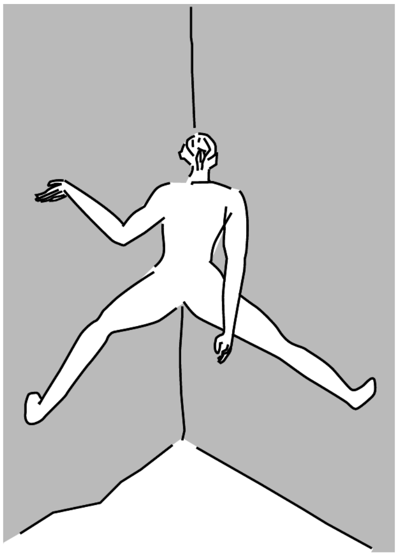
Problem 5-d9.

is desired and because the coefficient could be much lower if, for example, the surface was sandy rather than clean. We will assume that there is no friction where the rope goes over the lip of the cliff, although in reality this friction significantly reduces the load on the boulder.
(a) Let m be the mass of the climber, V the volume of the boulder, ρ its density, and g the strength of the gravitational field. Find the minimum value of μ_s. √
(b) Show that the units of your answer make sense.
(c) Check that its dependence on the variables makes sense.
(d) Evaluate your result numerically. The volume of my refrigerator is about 0.7 m^3, the density of granite is about 2.7 g/cm^3, and standards bodies use a body mass of 80 kg for testing climbing equipment. √

5-d10 The figure shows a rock climber wedged into a dihedral or "open book" consisting of two vertical walls of rock at an angle θ relative to one another. This position can be maintained without any ledges or holds, simply by pressing the feet against the walls. The left hand is being used just for a little bit of balance. (a) Find the minimum coefficient of friction between the rubber climbing shoes and the rock. (b) Interpret the behavior of your expression at extreme values of θ. (c) Steven Won has done tabletop experiments using climbing shoes on the rough back side of a granite slab from a kitchen countertop, and has estimated $\mu_s = 1.17$. Find the corresponding maximum value of θ.

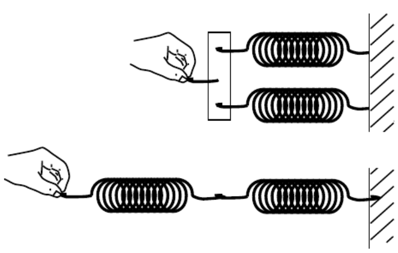
▷ Solution, p. 201

5-g1 The figure shows two different ways of combining a pair of identical springs, each with spring constant k. We refer to the top setup as parallel, and the bottom one as a series arrangement.
(a) For the parallel arrangement, analyze the forces acting on the connector piece on the left, and then use this analysis to determine the equivalent spring constant of the whole setup. Explain whether the combined spring constant should be interpreted as being stiffer or less stiff.



Problem 5-d10.

(b) For the series arrangement, analyze the forces acting on each spring and figure out the same things.



Problem 5-g1.

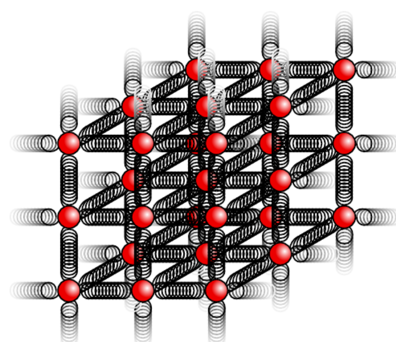
5-g2 Generalize the results of problem 5-g1 to the case where the two spring constants are unequal.

⋆

5-g3 (a) Using the solution of problem 5-g1, which is given in the back of the book, predict how the spring constant of a fiber will depend on its length and cross-sectional area.
(b) The constant of proportionality is called the Young's modulus, E, and typical values of the Young's modulus are about 10^{10} to 10^{11}. What units would the Young's modulus have in the SI (meter-kilogram-second) system?

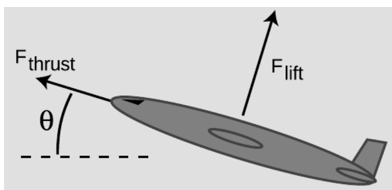
5-g4 This problem depends on the results of problems 5-g1 and 5-g3, whose solutions are in the back of the book. When atoms form chemical bonds, it makes sense to talk about the spring constant of the bond as a measure of how "stiff" it is. Of course, there aren't really little springs — this is just a mechanical model. The purpose of this problem is to estimate the spring constant, k, for a single bond in a typical piece of solid matter. Suppose we have a fiber, like a hair or a piece of fishing line, and imagine for simplicity that it is made of atoms of a single element stacked in a cubical manner, as shown in the figure, with a center-to-center spacing b. A typical value for b would be about 10^{-10} m.
(a) Find an equation for k in terms of b, and in terms of the Young's modulus, E, defined in problem 16 and its solution.
(b) Estimate k using the numerical data given in problem 5-g3.
(c) Suppose you could grab one of the atoms in a diatomic molecule like H_2 or O_2, and let the other atom hang vertically below it. Does the bond stretch by any appreciable fraction due to gravity?

⋆



Problem 5-g4.

5-j1 A cargo plane has taken off from a tiny airstrip in the Andes, and is climbing at constant speed, at an angle of $\theta = 17°$ with respect to horizontal. Its engines supply a thrust of $F_{\text{thrust}} = 200$ kN, and the lift from its wings is $F_{\text{lift}} = 654$ kN. Assume that air resistance (drag) is negligible, so the only forces acting are thrust, lift, and weight. What is its mass, in kg?



Problem 5-j1.

5-j2 A toy manufacturer is playtesting teflon booties that slip on over your shoes. In the parking lot, giggling engineers find that when they start with an initial speed of 1.2 m/s, they glide for 2.0 m before coming to a stop. What is the coefficient of friction between the asphalt and the booties?

✓

5-j3 A small piece of styrofoam packing material is dropped from rest at a height 2.00 m above the ground at time $t = 0$. The magnitude of its acceleration is given by $a = g - bv$, where v is the speed of the styrofoam, $g = 9.8\ \text{m/s}^2$, and b is a positive constant. After falling 0.500 m, the styrofoam effectively reaches terminal speed and then takes 5.00 s more to reach the ground.
(a) What is the acceleration (magnitude and direction) when $t = 0$? What about when the styrofoam is halfway to the ground?
(b) Find the terminal speed of the styrofoam. ✓
(c) What is the value of the constant b, with units? ✓
(d) What is the acceleration when the speed is 0.150 m/s? ✓
(e) Write $a = \mathrm{d}v/\mathrm{d}t$ and solve the differential equation for $v(t)$ (without plugging in numbers), and plot the result.

✓

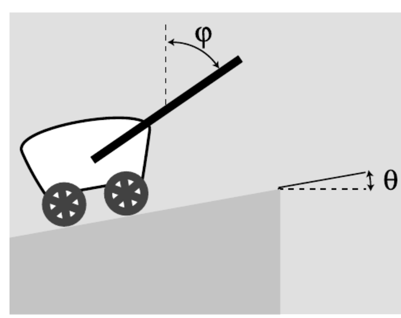
5-m1 Ice skaters with masses m_1 and m_2 push off from each other with a constant force F, which lasts until they lose contact. The distance between their centers of mass is ℓ_0 initially and ℓ_f when they lose contact.
(a) Find the amount of time T for which they remain in contact.
(b) Show that your answer in part a has units that make sense.
(c) Show that your answer has the right dependence on F.
(d) Interpret the case where one of the masses is very small.

▷ Solution, p. 201

5-m2 A wagon is being pulled at constant speed up a slope θ by a rope that makes an angle ϕ with the vertical.
(a) Assuming negligible friction, show that the tension in the rope is given by the equation

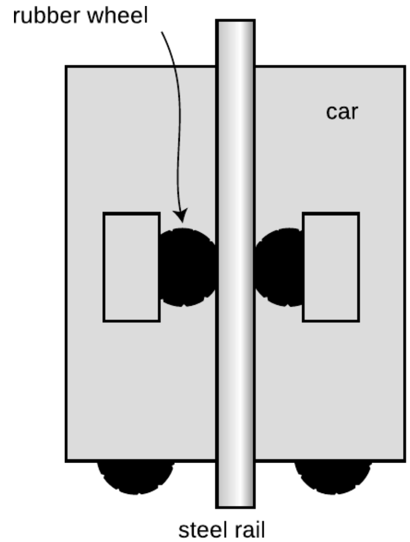
$$F_T = \frac{\sin\theta}{\sin(\theta+\phi)} F_W,$$

where F_W is the weight force acting on the wagon.
(b) Interpret this equation in the special cases of $\phi = 0$ and $\phi = 180° - \theta$.



Problem 5-m2.

5-m3 Today's tallest buildings are really not that much taller than the tallest buildings of the 1940's. One big problem with making an even taller skyscraper is that every elevator needs its own shaft running the whole height of the building. So many elevators are needed to serve the building's thousands of occupants that the elevator shafts start taking up too much of the space within the building. An alternative is to have elevators that can move both horizontally and vertically: with such a design, many elevator cars can share a few shafts, and they don't get in each other's way too much because they can detour around each other. In this design, it becomes impossible to hang the cars from cables, so they would instead have to ride on rails which they grab onto with wheels. Friction would keep them from slipping. The figure shows such a frictional elevator in its vertical travel mode. (The wheels on the bottom are for when it needs to switch to horizontal motion.)
(a) If the coefficient of static friction between rubber and steel is μ_s, and the maximum mass of the car plus its passengers is M, how much force must there be pressing each wheel against the rail in order to keep the car from slipping? (Assume the car is not accelerating.) ✓
(b) Show that your result has physically reasonable behavior with respect to μ_s. In other words, if there was less friction, would the wheels need to be pressed more firmly or less firmly? Does your equation behave that way?



Problem 5-m3.

5-m4 A skier of mass m is coasting down a slope inclined at an angle θ compared to horizontal. Assume for simplicity that the treatment of kinetic friction given in chapter 5 is appropriate here, although a soft and wet surface actually behaves a little differently. The coefficient of kinetic friction acting between the skis and the snow is μ_k, and in addition the skier experiences an air friction force of magnitude bv^2, where b is a constant.
(a) Find the maximum speed that the skier will attain, in terms of the variables m, g, θ, μ_k, and b. ✓
(b) For angles below a certain minimum angle θ_{min}, the equation gives a result that is not mathematically meaningful. Find an equation for θ_{min}, and give a physical explanation of what is happening for $\theta < \theta_{min}$. ✓

5-m5 Driving down a hill inclined at an angle θ with respect to horizontal, you slam on the brakes to keep from hitting a deer. Your antilock brakes kick in, and you don't skid.
(a) Analyze the forces. (Ignore rolling resistance and air friction.)
(b) Find the car's maximum possible deceleration, a (expressed as a positive number), in terms of g, θ, and the relevant coefficient of friction. ✓
(c) Explain physically why the car's mass has no effect on your answer.
(d) Discuss the mathematical behavior and physical interpretation of your result for negative values of θ.
(e) Do the same for very large positive values of θ.

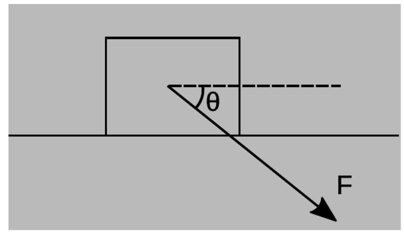
5-m6 An ice skater builds up some speed, and then coasts across the ice passively in a straight line. (a) Analyze the forces, using a table in the format shown in section 5.5.
(b) If his initial speed is v, and the coefficient of kinetic friction is μ_k, find the maximum theoretical distance he can glide before coming to a stop. Ignore air resistance. ✓
(c) Show that your answer to part b has the right units.
(d) Show that your answer to part b depends on the variables in a way that makes sense physically.
(e) Evaluate your answer numerically for $\mu_k = 0.0046$, and a world-record speed of 14.58 m/s. (The coefficient of friction was measured by De Koning et al., using special skates worn by real speed skaters.) ✓
(f) Comment on whether your answer in part e seems realistic. If it doesn't, suggest possible reasons why.

5-m7 A cop investigating the scene of an accident measures the length L of a car's skid marks in order to find out its speed v at the beginning of the skid. Express v in terms of L and any other relevant variables.

✓

5-m8 A force F is applied to a box of mass M at an angle θ below the horizontal (see figure). The coefficient of static friction between the box and the floor is μ_s, and the coefficient of kinetic friction between the two surfaces is μ_k.
(a) What is the magnitude of the normal force on the box from the floor? ✓
(b) What is the minimum value of F to get the box to start moving from rest? ✓
(c) What is the value of F so that the box will move with constant velocity (assuming it is already moving)? ✓
(d) If θ is greater than some critical angle θ_crit, it is impossible to have the scenario described in part c. What is θ_crit?

✓



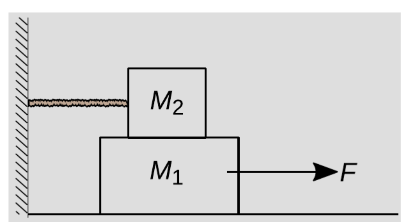
Problem 5-m8.

5-m9 A ramp of length L is inclined at angle θ to the horizontal. You would like to push your backpack, which has mass m, to the top of the ramp. The coefficient of kinetic friction between the backpack and the ramp is μ_k, and you will push the backpack with a force of magnitude F.
(a) How long will it take the backpack to reach the top of the ramp if you apply your push parallel to the ramp's surface? ✓
(b) How long will it take the backpack to reach the top of the ramp if you apply your push parallel to the the ground?

✓

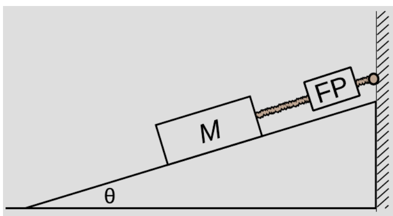
5-m10 Blocks M_1 and M_2 are stacked as shown, with M_2 on top. M_2 is connected by a string to the wall, and M_1 is pulled to the right with a force F big enough to get M_1 to move. The coefficient of kinetic friction has the same value μ_k among all surfaces (i.e., the block-block and ground-block interfaces).
(a) Analyze the forces in which each block participates, as in section 5.5.
(b) Determine the tension in the string. ✓
(c) Find the acceleration of the block of mass M_1.

✓



Problem 5-m10.

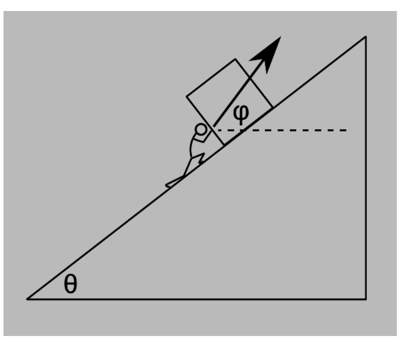
5-m11 (a) A mass M is at rest on a fixed, frictionless ramp inclined at angle θ with respect to the horizontal. The mass is connected to the force probe, as shown. What is the reading on the force probe? ✓
(b) Check that your answer to part a makes sense in the special cases $\theta = 0$ and $\theta = 90°$.



Problem 5-m11.

5-m12 You are pushing a box up a ramp that is at an angle θ with respect to the horizontal. Friction acts between the box and the ramp, with coefficient μ. Suppose that your force is fixed in magnitude, but can be applied at any desired angle φ above the horizontal. Find the optimal value of φ.

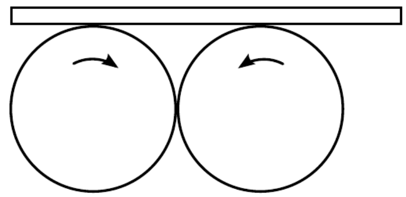
✓ ⋆



Problem 5-m12.

5-m13 Two wheels of radius r rotate in the same vertical plane with angular velocities $+\Omega$ and $-\Omega$ (rates of rotation in radians per second) about axes that are parallel and at the same height. The wheels touch one another at a point on their circumferences, so that their rotations mesh like gears in a gear train. A board is laid on top of the wheels, so that two friction forces act upon it, one from each wheel. Characterize the three qualitatively different types of motion that the board can exhibit, depending on the initial conditions.

⋆⋆

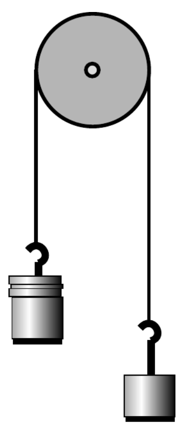


Problem 5-m13.

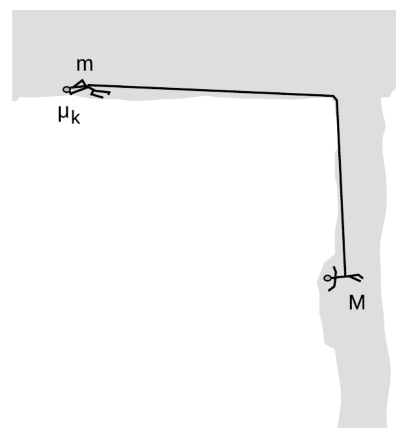
5-p1 A tugboat of mass m pulls a ship of mass M, accelerating it. The speeds are low enough that you can ignore fluid friction acting on their hulls, although there will of course need to be fluid friction acting on the tug's propellers.
(a) Analyze the forces in which the tugboat participates, using a table in the format shown in section 5.5. Don't worry about vertical forces.
(b) Do the same for the ship.
(c) If the force acting on the tug's propeller is F, what is the tension, T, in the cable connecting the two ships? [Hint: Write down two equations, one for Newton's second law applied to each object. Solve these for the two unknowns T and a.] ✓
(d) Interpret your answer in the special cases of $M = 0$ and $M = \infty$.

5-p2 Unequal masses M and m are suspended from a pulley as shown in the figure.
(a) Analyze the forces in which mass m participates, using a table in the format shown in section 5.5. [The forces in which the other mass participates will of course be similar, but not numerically the same.]
(b) Find the magnitude of the accelerations of the two masses. [Hints: (1) Pick a coordinate system, and use positive and negative signs consistently to indicate the directions of the forces

and accelerations. (2) The two accelerations of the two masses have to be equal in magnitude but of opposite signs, since one side eats up rope at the same rate at which the other side pays it out. (3) You need to apply Newton's second law twice, once to each mass, and then solve the two equations for the unknowns: the acceleration, a, and the tension in the rope, T.] √
(c) Many people expect that in the special case of $M = m$, the two masses will naturally settle down to an equilibrium position side by side. Based on your answer from part b, is this correct?
(d) Find the tension in the rope, T. √
(e) Interpret your equation from part d in the special case where one of the masses is zero. Here "interpret" means to figure out what happens mathematically, figure out what should happen physically, and connect the two.

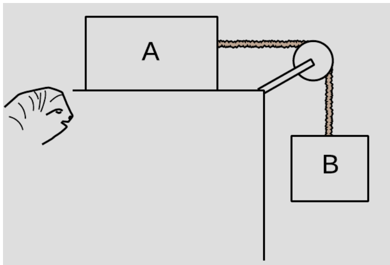

Problem 5-p2.

5-p3 Mountain climbers with masses m and M are roped together while crossing a horizontal glacier when a vertical crevasse opens up under the climber with mass M. The climber with mass m drops down on the snow and tries to stop by digging into the snow with the pick of an ice ax. Alas, this story does not have a happy ending, because this doesn't provide enough friction to stop. Both m and M continue accelerating, with M dropping down into the crevasse and m being dragged across the snow, slowed only by the kinetic friction with coefficient μ_k acting between the ax and the snow. There is no significant friction between the rope and the lip of the crevasse.
(a) Find the acceleration a. √
(b) Check the units of your result.
(c) Check the dependence of your equation on the variables. That means that for each variable, you should determine what its effect on a should be physically, and then what your answer from part a says its effect would be mathematically.

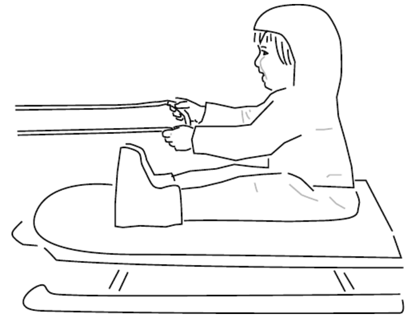


Problem 5-p3.

5-p4 Consider the system shown in the figure. Block A has mass M_A and block B has mass M_B. There is friction between the table and block A. Once block B is set into downward motion, it descends at a constant speed.
(a) Analyze the forces in which each block participates as described in section 5.5
(b) Calculate the coefficient of kinetic friction between block A and the tabletop. √
(c) A sloth, also of mass M_A, falls asleep on top of block A. If block B is now set into downward motion, what is its acceleration (magnitude and direction)?
√



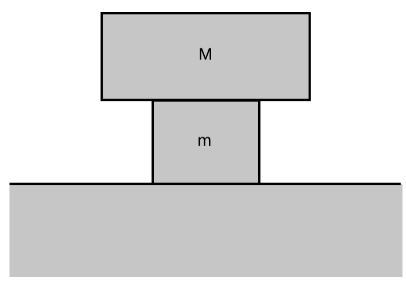
Problem 5-p4.


Problem 5-p5.

5-p5 Ginny has a plan. She is going to ride her sled while her dog Foo pulls her, and she holds on to his leash. However, Ginny hasn't taken physics, so there may be a problem: she may slide right off the sled when Foo starts pulling.
(a) Analyze all the forces in which Ginny participates, making a table as in section 5.5.
(b) Analyze all the forces in which the sled participates.
(c) The sled has mass m, and Ginny has mass M. The coefficient of static friction between the sled and the snow is μ_1, and μ_2 is the corresponding quantity for static friction between the sled and her snow pants. Ginny must have a certain minimum mass so that she will not slip off the sled. Find this in terms of the other three variables. ✓
(d) Interpreting your equation from part c, under what conditions will there be no physically realistic solution for M? Discuss what this means physically.

5-p6 The figure shows a stack of two blocks, sitting on top of a table that is bolted to the floor. All three objects are made from identical wood, with their surfaces finished identically using the same sandpaper. We tap the middle block, giving it an initial velocity v to the right. The tap is executed so rapidly that almost no initial velocity is imparted to the top block.
(a) Find the time that will elapse until the slipping between the top and middle blocks stops. Express your answer in terms of v, m, M, g, and the relevant coefficient of friction. ✓
(b) Show that your answer makes sense in terms of units.
(c) Check that your result has the correct behavior when you make m bigger or smaller. Explain. This means that you should discuss the mathematical behavior of the result, and then explain how this corresponds to what would really happen physically.
(d) Similarly, discuss what happens when you make M bigger or smaller.
(e) Similarly, discuss what happens when you make g bigger or smaller.

⋆

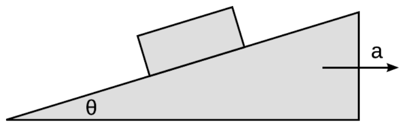


Problem 5-p6.

5-p7 (a) A block is sitting on a wedge inclined at an angle θ with respect to horizontal. Someone grabs the wedge and moves it horizontally with acceleration a. The motion is in the direction shown by the arrow in the figure. Find the maximum acceleration that can be applied

without causing the block to slide downhill. ✓
(b) Show that your answer to part a has the right units.
(c) Show that it also has the right dependence on θ, by comparing its mathematical behavior to its physically expected behavior.

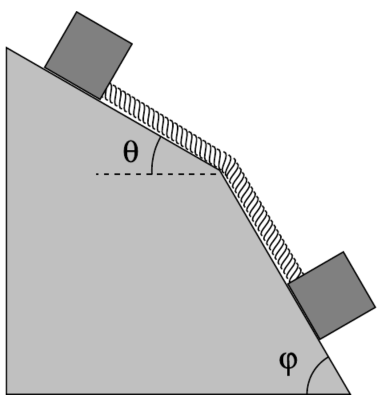
⋆



Problem 5-p7.

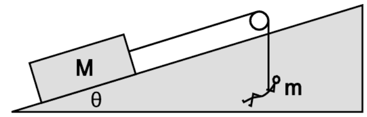
5-p8 The two blocks shown in the figure have equal mass, m, and the surface is frictionless. (a) What is the tension in the massless rope? ✓
(b) Show that the units of your answer make sense.
(c) Check the physical behavior of your answer in the special cases of $\phi \leq \theta$ and $\theta = 0$, $\phi = 90°$.

⋆



Problem 5-p8.

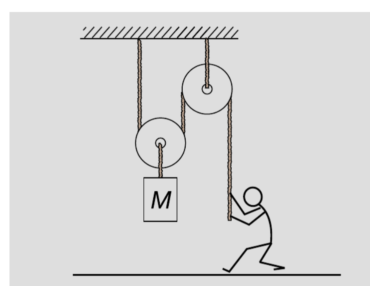
5-p9 (a) The person with mass m hangs from the rope, hauling the box of mass M up a slope inclined at an angle θ. There is friction between the box and the slope, described by the usual coefficients of friction. The pulley, however, is frictionless. Find the magnitude of the box's acceleration. ✓
(b) Show that the units of your answer make sense.
(c) Check the physical behavior of your answer in the special cases of $M = 0$ and $\theta = -90°$.



Problems 5-p9 and 5-p10.

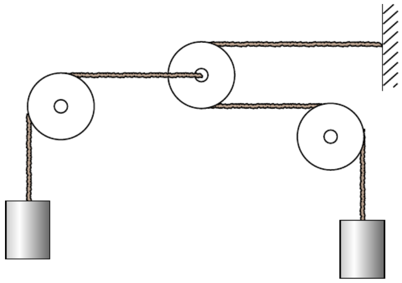
5-p10 The physical situation is the same as in problem 5-p9, except that we make different assumptions about friction. (a) Suppose that there is no friction between the block and the ramp. Find the value of m/M so that the system is in equilibrium. ✓
(b) If there is instead a coefficient of static friction μ_s between the block and the ramp, find the minimum and maximum values that m/M can have so that the blocks remain at rest. ✓

5-s1 A person can pull with a maximum force F. What is the maximum mass that the person can lift with the pulley setup shown in the figure? ✓



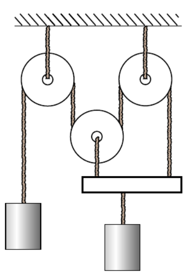
Problem 5-s1.

5-s2 In the system shown in the figure, the pulleys on the left and right are fixed, but the pulley in the center can move to the left or right. The two masses are identical. Find the upward acceleration of the mass on the left, in terms of g only. Assume all the ropes and pulleys are massless and frictionless.



Problem 5-s2.

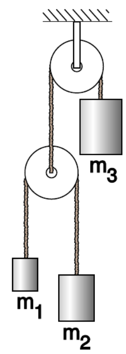
5-s3 The two masses are identical. Find the upward acceleration of the mass on the right, in terms of g only. Assume all the ropes and pulleys, as well as the cross-bar, are massless, and the pulleys are frictionless. The right-hand mass has been positioned away from the bar's center, so that the bar will not twist.



Problem 5-s3.

5-s4 Find the upward acceleration of mass m_1 in the figure.

√



Problem 5-s4.

6 Circular motion

This is not a textbook. It's a book of problems meant to be used along with a textbook. Although each chapter of this book starts with a brief summary of the relevant physics, that summary is not meant to be enough to allow the reader to actually learn the subject from scratch. The purpose of the summary is to show what material is needed in order to do the problems, and to show what terminology and notation are being used.

6.1 Uniform circular motion

Figure 6.1 shows an overhead view of a person swinging a rock on a rope. A force from the string is required to make the rock's velocity vector keep changing direction. If the string breaks, the rock will follow Newton's first law and go straight instead of continuing around the circle. Circular motion requires a force with a component toward the center of the circle.

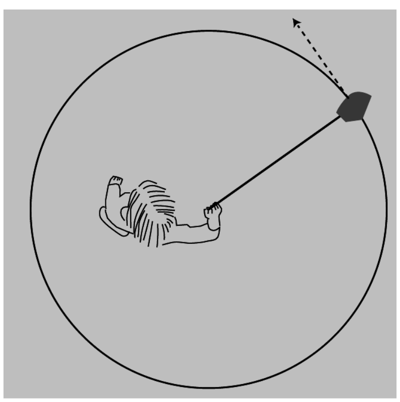


Figure 6.1: Overhead view of a person swinging a rock on a rope.

Uniform circular motion is the special case in which the speed is constant. In uniform circular motion, the acceleration vector is toward the center, and therefore total force acting on the object must point *directly* toward the center. We can define the *angular velocity* ω, which is the number of radians per second by which the object's angle changes, $\omega = \mathrm{d}\theta/\,\mathrm{d}t$. For uniform circular motion, ω is constant, and the magnitude of the acceleration is

$$a = \omega^2 r = \frac{v^2}{r}, \qquad (6.1)$$

where r is the radius of the circle (see problem 6-d1, p. 72). These expressions can also be related to the *period* of the rotation T, which is the time for one revolution. We have $\omega = 2\pi/T$.

6.2 Rotating frames

When you're in the back seat of a car going around a curve, not looking out the window, there is a strong tendency to adopt a frame of reference in which the car is at rest. This is a noninertial frame of reference, because the car is accelerating. In a noninertial frame, Newton's laws are violated. For example, the air freshener hanging from the mirror in figure 3.6 on p. 33 will swing as the car enters the curve, but this motion is not caused by a force made by any identifiable object. In the inertial frame of someone standing by the side of the road, the air freshener simply continued straight while the *car* accelerated.

6.3 Nonuniform motion

In nonuniform circular motion, we have not just the acceleration $a_r = \omega^2 r$ in the radial direction (toward the center of the circle) but also a component $a_t = \mathrm{d}v/\,\mathrm{d}t$ in the tangential direction.

6.4 Rotational kinematics

Angular velocity and acceleration

If a rigid body such as a top rotates about a fixed axis, then every particle in that body per-

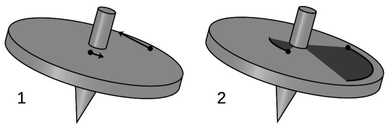


Figure 6.2: Different atoms in the top have different velocity vectors, 1, but sweep out the same angle in a given time, 2.

forms circular motion about a point on that axis. Every atom has a different velocity vector, figure 6.2. Since all the velocities are different, we can't measure the top's speed of rotation of the top by giving a single velocity. But every particle covers the same angle in the same amount of time, so we can specify the speed of rotation consistently in terms of angle per unit time. Let the position of some reference point on the top be denoted by its angle θ, measured in a circle around the axis. We measure all our angles in radians. We define the angular velocity as

$$\omega = \frac{\mathrm{d}\theta}{\mathrm{d}t}.$$

The relationship between ω and t is exactly analogous to that between x and t for the motion of a particle through space. The angular velocity has units of radians per second, s^{-1}. We also define an angular acceleration,

$$\alpha = \frac{\mathrm{d}\omega}{\mathrm{d}t}.$$

with units s^{-2}.

The mathematical relationship between ω and θ is the same as the one between v and x, and similarly for α and a. We can thus make a system of analogies, and recycle all the familiar kinematic equations for constant-acceleration motion.

Angular and linear quantities related

We often want to relate the angular quantities to the motion of a particular point on the rotating

$$\mathbf{x} \longleftrightarrow \theta$$

$$\mathbf{v} \longleftrightarrow \omega$$

$$\mathbf{a} \longleftrightarrow \alpha$$

Figure 6.3: Analogies between rotational and linear quantities.

object. The velocity vector has tangential and radial components

$$v_t = \omega r$$

and

$$v_r = 0.$$

For the acceleration vector,

$$a_t = \alpha r$$

and

$$a_r = \omega^2 r.$$

Problems

6-a1 Show that the expression $|\mathbf{v}|^2/r$ has the units of acceleration.

6-a2 A plane is flown in a loop-the-loop of radius 1.00 km. The plane starts out flying upside-down, straight and level, then begins curving up along the circular loop, and is right-side up when it reaches the top. (The plane may slow down somewhat on the way up.) How fast must the plane be going at the top if the pilot is to experience no force from the seat or the seatbelt while at the top of the loop?
$\checkmark$

6-a3 The bright star Sirius has a mass of 4.02×10^{30} kg and lies at a distance of 8.1×10^{16} m from our solar system. Suppose you're standing on a merry-go-round carousel rotating with a period of 10 seconds, and Sirius is on the horizon. You adopt a rotating, noninertial frame of reference, in which the carousel is at rest, and the universe is spinning around it. If you drop a corndog, you see it accelerate horizontally away from the axis, and you interpret this as the result of some horizontal force. This force does not actually exist; it only seems to exist because you're insisting on using a noninertial frame. Similarly, calculate the force that seems to act on Sirius in this frame of reference. Comment on the physical plausibility of this force, and on what object could be exerting it.
$\checkmark$

6-a4 Lionel brand toy trains come with sections of track in standard lengths and shapes. For circular arcs, the most commonly used sections have diameters of 662 and 1067 mm at the inside of the outer rail. The maximum speed at which a train can take the broader curve without flying off the tracks is 0.95 m/s. At what speed must the train be operated to avoid derailing on the tighter curve?
$\checkmark$

6-a5 Debbie is in Los Angeles, at a distance $r = 5280$ km from the Earth's axis of rotation.
(a) What is Debbie's speed (as measured by an observer not moving relative to the center of the Earth)? $\checkmark$
(b) What is Debbie's acceleration? Your answer indicates to what extent the apparent acceleration due to gravity is modified when not at one of the poles. For reference, $g = 9.81$ m/s^2 at the poles, but appears slightly less near the equator.
$\checkmark$

6-a6 Some kids are playing around on a merry-go-round at the park. You notice that little Timmy can't seem to hold on to the edge of the merry-go-round (of radius 1.5 m) when the period of the merry-go-round (the time for one full revolution) is less than 2.0 s.
(a) What is the maximum acceleration that Timmy can handle? $\checkmark$
(b) If Timmy were on a merry-go-round of radius 6.0 m, what wold be the minimum period of revolution that would allow him to stay on?
$\checkmark$

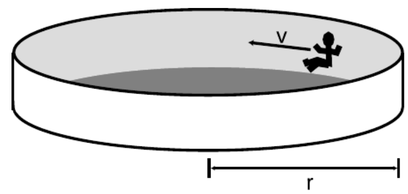
6-a7 A particle is undergoing uniform circular motion in the xy-plane such that its distance to the origin does not change. At one instant, the particle is moving with velocity $\mathbf{v} = (10.0\ \text{m/s})\hat{\mathbf{x}}$ and with acceleration $\mathbf{a} = (2.0\ \text{m/s}^2)\hat{\mathbf{y}}$.
(a) What is the radius of the circle? $\checkmark$
(b) How long does it take for the particle to travel once around the circle (i.e., what is the period of motion)? $\checkmark$
(c) In what direction will the particle be moving after a quarter of a period? Give your answer as an angle θ $(0 \leq \theta < 360°)$ that the velocity vector makes with respect to the $+\hat{\mathbf{x}}$ direction, measured counterclockwise from the $+\hat{\mathbf{x}}$ direction.
$\checkmark$

6-a8 A car is approaching the top of a hill of radius of curvature R.
(a) If the normal force that the driver feels at the top of the hill is 1/3 of their weight, how fast is the car going? $\checkmark$
(b) Check that the units of your answer to part a make sense.
(c) Check that the dependence of your answer on the variables makes sense.

6-a9 An airplane is in a nosedive. In order not to crash, the pilot pulls up as much as she can. At the lowest point of the plane's trajectory, the plane is in a circular arc of radius 300 m, and the pilot experiences an acceleration of $5.0g$ (so that her weight feels like six times normal). What is the plane's speed at this point? ✓

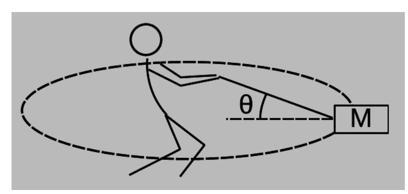
6-d1 In this problem, you'll derive the equation $|\mathbf{a}| = |\mathbf{v}|^2/r$ using calculus. Instead of comparing velocities at two points in the particle's motion and then taking a limit where the points are close together, you'll just take derivatives. The particle's position vector is $\mathbf{r} = (r\cos\theta)\hat{\mathbf{x}} + (r\sin\theta)\hat{\mathbf{y}}$, where $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are the unit vectors along the x and y axes. By the definition of radians, the distance traveled since $t = 0$ is $r\theta$, so if the particle is traveling at constant speed $v = |\mathbf{v}|$, we have $v = r\theta/\mathrm{t}$.
(a) Eliminate θ to get the particle's position vector as a function of time.
(b) Find the particle's acceleration vector.
(c) Show that the magnitude of the acceleration vector equals v^2/r.

6-g1 The amusement park ride shown in the figure consists of a cylindrical room that rotates about its vertical axis. When the rotation is fast enough, a person against the wall can pick his or her feet up off the floor and remain "stuck" to the wall without falling.
(a) Suppose the rotation results in the person having a speed v. The radius of the cylinder is r, the person's mass is m, the downward acceleration of gravity is g, and the coefficient of static friction between the person and the wall is μ_s. Find an equation for the speed, v, required, in terms of the other variables. (You will find that one of the variables cancels out.)
(b) Now suppose two people are riding the ride. Huy is wearing denim, and Gina is wearing polyester, so Huy's coefficient of static friction is three times greater. The ride starts from rest, and as it begins rotating faster and faster, Gina must wait longer before being able to lift her feet without sliding to the floor. Based on your equation from part a, how many times greater must the speed be before Gina can lift her feet without sliding down?



Problem 6-g1.

6-g2 Tommy the playground bully is whirling a brick tied to the end of a rope. The rope makes an angle θ with respect to the horizontal, and the brick undergoes circular motion with radius R.
(a) What is the speed of the brick? ✓
(b) Check that the units of your answer to part a make sense.
(c) Check that the dependence of your answer on the variables makes sense, and comment on the limit $\theta \to 0$.

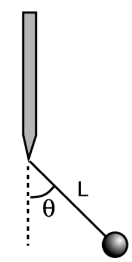


Problem 6-g2.

6-g3 The figure shows a ball on the end of a string of length L attached to a vertical rod which is spun about its vertical axis by a motor. The period (time for one rotation) is P.
(a) Analyze the forces in which the ball participates.
(b) Find how the angle θ depends on P, g, and L. [Hints: (1) Write down Newton's second law for the vertical and horizontal components of

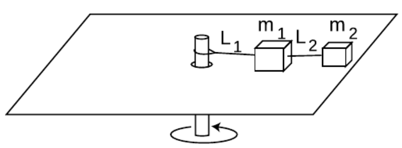
force and acceleration. This gives two equations, which can be solved for the two unknowns, θ and the tension in the string. (2) If you introduce variables like v and r, relate them to the variables your solution is supposed to contain, and eliminate them.] ✓

(c) What happens mathematically to your solution if the motor is run very slowly (very large values of P)? Physically, what do you think would actually happen in this case?



Problem 6-g3.

6-g4 The figure shows two blocks of masses m_1 and m_2 sliding in circles on a frictionless table. Find the tension in the strings if the period of rotation (time required for one rotation) is P. ✓



Problem 6-g4.

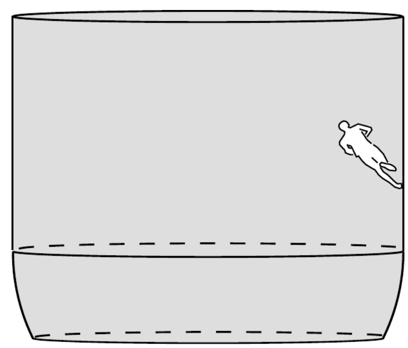
6-g5 In a well known stunt from circuses and carnivals, a motorcyclist rides around inside a big bowl, gradually speeding up and rising higher. Eventually the cyclist can get up to where the walls of the bowl are vertical. Let's estimate the conditions under which a running human could do the same thing.

(a) If the runner can run at speed v, and her shoes have a coefficient of static friction μ_s, what is the maximum radius of the circle? ✓

(b) Show that the units of your answer make sense.

(c) Check that its dependence on the variables makes sense.

(d) Evaluate your result numerically for $v = 10$ m/s (the speed of an olympic sprinter) and $\mu_s = 5$. (This is roughly the highest coefficient of static friction ever achieved for surfaces that are not sticky. The surface has an array of microscopic fibers like a hair brush, and is inspired by the hairs on the feet of a gecko. These assumptions are not necessarily realistic, since the person would have to run at an angle, which would be physically awkward.) ✓



Problem 6-g5.

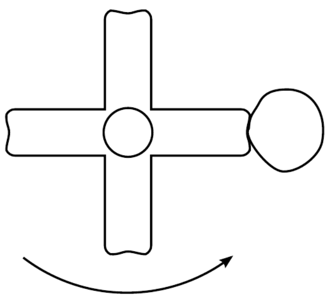
6-g6 A child places a toy on the outer rim of a merry-go-round that has radius R and period T. What is the lowest value of the coefficient of static friction between the toy and the merry-go-round that allows the toy to stay on the carousel? ✓

6-j1 When you're done using an electric mixer, you can get most of the batter off of the beaters by lifting them out of the batter with the motor running at a high enough speed. Let's imagine, to make things easier to visualize, that we instead have a piece of tape stuck to one of the beaters.

(a) Explain why static friction has no effect on whether or not the tape flies off.

(b) Analyze the forces in which the tape participates, using a table in the format shown in section 5.5.
(c) Suppose you find that the tape doesn't fly off when the motor is on a low speed, but at a greater speed, the tape won't stay on. Why would the greater speed change things? [Hint: If you don't invoke any law of physics, you haven't explained it.]

⋆



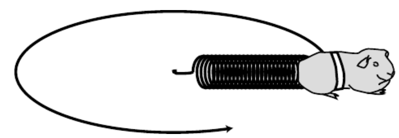
Problem 6-j1.

6-j2 The acceleration of an object in uniform circular motion can be given either by $|\mathbf{a}| = |\mathbf{v}|^2/r$ or, equivalently, by $|\mathbf{a}| = 4\pi^2 r/T^2$, where T is the time required for one cycle. Person A says based on the first equation that the acceleration in circular motion is greater when the circle is smaller. Person B, arguing from the second equation, says that the acceleration is smaller when the circle is smaller. Rewrite the two statements so that they are less misleading, eliminating the supposed paradox. [Based on a problem by Arnold Arons.]

⋆

6-j3 Psychology professor R.O. Dent requests funding for an experiment on compulsive thrill-seeking behavior in guinea pigs, in which the subject is to be attached to the end of a spring and whirled around in a horizontal circle. The spring has relaxed length b, and obeys Hooke's law with spring constant k. It is stiff enough to keep from bending significantly under the guinea pig's weight.
(a) Calculate the length of the spring when it is undergoing steady circular motion in which one rotation takes a time T. Express your result in terms of k, b, T, and the guinea pig's mass m. ✓
(b) The ethics committee somehow fails to veto the experiment, but the safety committee expresses concern. Why? Does your equation do anything unusual, or even spectacular, for any particular value of T? What do you think is the physical significance of this mathematical behavior?

⋆



Problem 6-j3.

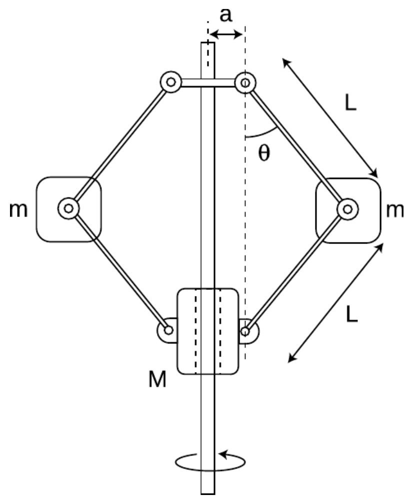
6-j4 The figure shows an old-fashioned device called a flyball governor, used for keeping an engine running at the correct speed. The whole thing rotates about the vertical shaft, and the mass M is free to slide up and down. This mass would have a connection (not shown) to a valve that controlled the engine. If, for instance, the engine ran too fast, the mass would rise, causing the engine to slow back down.
(a) Show that in the special case of $a = 0$, the angle θ is given by

$$\theta = \cos^{-1}\left(\frac{g(m+M)P^2}{4\pi^2 mL}\right),$$

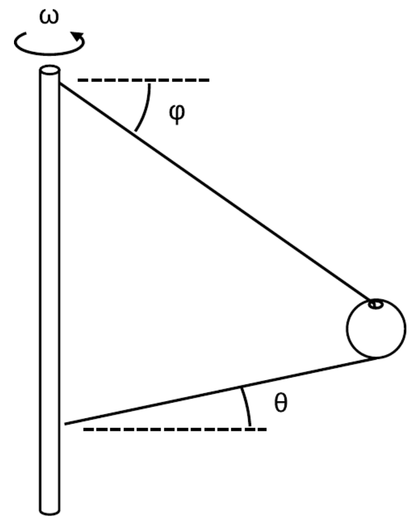
where P is the period of rotation (time required for one complete rotation).
(b) There is no closed-form solution for θ in the general case where a is not zero. However, explain how the undesirable low-speed behavior of the $a = 0$ device would be improved by making a nonzero.

⋆

6-j5 A car exits the freeway via a circular off-ramp. The road is level (i.e., the curve is not banked at all), and the radius of curvature of the circular ramp is 1.00×10^2 m. The coefficient of



Problem 6-j4.



Problem 6-j6.

friction between the tires and the road is $\mu = 0.8$.
(a) What is the maximum possible speed the car can travel on the offramp without slipping? ✓
For parts (b) through (d), the car is traveling at 20.0 m/s and decelerating at a rate of 3.0 m/s^2. The car is turning to the right.
(b) What is the absolute value of the radial component of the acceleration? What is the magnitude of the acceleration vector? ✓
(c) There is a pine-scented tree hanging from the rear view mirror. During the curve, the tree does not hang vertically. Describe in words which way the tree swings. Does it appear to lean left, or right? Forward, or backward? An analysis of the forces might help in your explanation.
(d) What angle does the tree make with respect to the vertical?

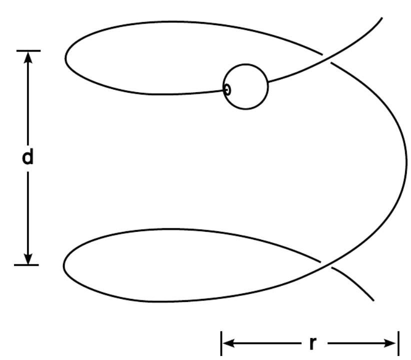
✓ ⋆

6-j6 The vertical post rotates at frequency ω. The bead slides freely along the string, reaching an equilibrium in which its distance from the axis is r and the angles θ and ϕ have some particular values. Find ϕ in terms of θ, g, ω, and r.

✓ ⋆

6-j7 A bead slides down along a piece of wire that is in the shape of a helix. The helix lies on the surface of a vertical cylinder of radius r, and the vertical distance between turns is d.
(a) Ordinarily when an object slides downhill under the influence of kinetic friction, the velocity-independence of kinetic friction implies that the acceleration is constant, and therefore there is no limit to the object's velocity. Explain the physical reason why this argument fails here, so that the bead will in fact have some limiting velocity.
(b) Find the limiting velocity.
(c) Show that your result has the correct behavior in the limit of $r \to \infty$. [Problem by B. Korsunsky.]

✓ ⋆⋆



Problem 6-j7.

6-m1 A disk, initially rotating at 120 radians per second, is slowed down with a constant angular acceleration of magnitude 4.0 s^{-2}. How many revolutions does the disk make before it comes to rest?

$\surd$

6-m2 A bell rings at the Tilden Park merry go round in Berkeley, California, and the carousel begins to move with an angular acceleration of 1.0×10^{-2} s^{-2}. How much time does it take to perform its first revolution?

$\surd$

6-m3 Neutron stars are the collapsed remnants of dead stars. They rotate quickly, and their rotation can be measured extremely accurately by radio astronomers. Some of them rotate at such a predictable rate that they can be used to count time about as accurately as the best atomic clocks. They do decelerate slowly, but this deceleration can be taken into account. One of the best-studied stars of this type[1] was observed continuously over a 10-year period. As of the benchmark date April 5, 2001, it was found to have

$$\omega = 1.091313551502333 \times 10^3\ \text{s}^{-1}$$

and

$$\alpha = -1.085991 \times 10^{-14}\ \text{s}^{-2},$$

where the error bars in the final digit of each number are about ± 1. Astronomers often use the Julian year as their unit of time, where one Julian year is defined to be exactly 3.15576×10^7 s. Find the number of revolutions that this pulsar made over a period of 10 Julian years, starting from the benchmark date.

$\surd$

6-m4 A gasoline-powered car has a heavy wheel called a flywheel, whose main function is to add inertia to the motion of the engine so that it keeps spinning smoothly between power strokes of the cylinders. Suppose that a certain car's flywheel is spinning with angular velocity ω_0, but the car is then turned off, so that the engine and flywheel start to slow down as a result of friction. Assume that the angular acceleration is constant. After the flywheel has made N revolutions, it comes to rest. What is the magnitude of the angular acceleration?

$\surd$

6-m5 A rigid body rotates about a line according to $\theta = At^3 - Bt$ (valid for both negative and positive t).
(a) What is the angular velocity as a function of time? $\surd$
(b) What is the angular acceleration as a function of time? $\surd$
(c) There are two times when the angular velocity is zero. What is the positive time for which this is true? Call this t_+. $\surd$
(d) What is the average angular velocity over the time interval from 0 to t_+?

$\surd$

6-m6 The angular acceleration of a wheel is $\alpha = 12t - 24t^3$, where α is in s^{-2} and t is the time in seconds. The wheel starts from rest at $t = 0$. How many revolutions has it turned between $t = 0$ and when it is again at rest?

$\surd$

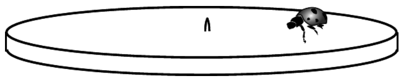
6-p1 (a) Find the angular velocities of the earth's rotation and of the earth's motion around the sun. $\surd$
(b) Which motion involves the greater acceleration?

6-p2 A bug stands on a horizontal turntable at distance r from the center. The coefficient of static friction between the bug and the turntable is μ_s. The turntable spins at constant angular frequency ω.
(a) Is the bug more likely to slip at small values of r, or large values?
(b) If the bug walks along a radius, what is the value of r at which it looses its footing?

$\surd$

6-p3 A bug stands on a horizontal turntable at distance r from the center. The coefficient of static friction between the bug and the turntable is μ_s. Starting from rest, the turntable begins

[1] Verbiest *et al.*, Astrophysical Journal 679 (675) 2008



Problems 6-p2, 6-p3, and 6-p4.

rotating with angular acceleration α. What is the magnitude of the angular frequency at which the bug starts to slide?

$\checkmark$

6-p4 A 20.0 g cockroach is lounging at a distance $r = 5.00$ cm from the axis of the carousel of a microwave oven. Except for the species, the situation is similar to the one shown in the figure. The cockroach's angular coordinate is $\theta(t) = 12t^2 - 4.0t^3$, where t is in seconds and θ is in radians.
(a) Find the angular velocity and acceleration. $\checkmark$
(b) At what time $t_2 > 0$ is the cockroach at rest? At what time t_1, where $0 < t_1 < t_2$, is the cockroach moving the fastest? $\checkmark$
(c) Find the tangential and radial components of the *linear* acceleration of the cockroach at t_1. What is the direction of the acceleration vector at this time? $\checkmark$
(d) How many revolutions does the cockroach make from $t = 0$ to $t = t_2$? $\checkmark$
(e) Suppose that the carousel, instead of decelerating, had kept spinning at constant speed after t_1. Find the period and the frequency (in revolutions per minute).

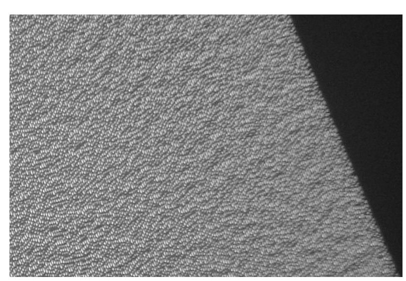
$\checkmark$

6-p5 A CD is initially moving counterclockwise with angular speed ω_0 and then starts decelerating with an angular acceleration of magnitude α.
(a) How long does it take for the CD to come to rest? $\checkmark$
(b) Suppose that, after a time equal to half that found in part a, point on the CD satisfies $|a_r| = |a_t|$. If the initial angular velocity of the CD was 40 s^{-1}, what is the total time it takes for the CD to come to rest (i.e., the numerical value of your answer in part a)?

$\checkmark$

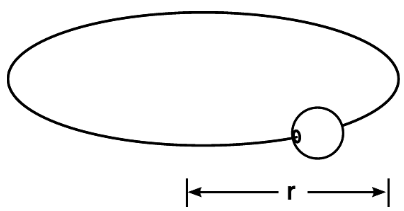
6-s1 The figure shows a microscopic view of the innermost tracks of a music CD. The pits represent the pattern of ones and zeroes that encode the musical waveform. Because the laser that reads the data has to sweep over a fixed amount of data per unit time, the disc spins at a decreasing angular velocity as the music is played from the inside out. The linear velocity v, not the angular velocity, is constant. Each track is separated from its neighbors on either side by a fixed distance p, called the pitch. Although the tracks are actually concentric circles, we will idealize them in this problem as a type of spiral, called an Archimedean spiral, whose turns have constant spacing, p, along any radial line. Our goal is to find the angular acceleration of this idealized CD, in terms of the constants v and p, and the radius r at which the laser is positioned.
(a) Use geometrical reasoning to constrain the dependence of the result on p.
(b) Use units to further constrain the result up to a unitless multiplicative constant.
(c) Find the full result. [Hint: Find a differential equation involving r and its time derivative, and then solve this equation by separating variables.] $\checkmark$
(d) Consider the signs of the variables in your answer to part c, and show that your equation still makes sense when the direction of rotation is reversed.
(e) Similarly, check that your result makes sense regardless of whether we view the CD player from the front or the back. (Clockwise seen from one side is counterclockwise from the other.)

$\star$



Problem 6-s1.

6-s2 Find the motion of a bead that slides with coefficient of kinetic friction μ on a circular wire of radius r. Neglect gravity. [This requires a couple of standard techniques for solving a differential equation, but not obscure or tricky ones.] $\star$



Problem 6-s2.

7 Conservation of energy

This is not a textbook. It's a book of problems meant to be used along with a textbook. Although each chapter of this book starts with a brief summary of the relevant physics, that summary is not meant to be enough to allow the reader to actually learn the subject from scratch. The purpose of the summary is to show what material is needed in order to do the problems, and to show what terminology and notation are being used.

7.1 Conservation laws

Newton presented his laws of motion as universal ones that would apply to all phenomena. We now know that this is not true. For example, a ray of light has zero mass, so $a = F/m$ gives nonsense when applied to light. Today, physicists formulate the most fundamental laws of physics as *conservation laws*, which arise from *symmetry* principles.

An object has a symmetry if it remains unchanged under some sort of transformation such as a reflection, rotation, rotation, or translation in time or space. A sphere is symmetric under rotation. An object that doesn't change over time has symmetry with respect to time-translation.

Figure 7.1: In this scene from Swan Lake, the choreography has a symmetry with respect to left and right.

The fundamentally important symmetries in physics are not symmetries of objects but symmetries of the laws of physics themselves. One such symmetry is that laws of physics do not seem to change over time. That is, they have time-translation symmetry. The gravitational forces that you see near the surface of the earth are determined by Newton's law of gravity, which we will state later in quantitative detail.

Suppose that Newton's law of gravity *did* change over time. (Such a change would have to be small, because precise experiments haven't shown objects to get heavier or lighter from one time to another.) If you knew of such a change, then you could exploit it to make money. On a day when gravity was weak, you could pay the electric company what it cost you to run an electric motor, and lift a giant weight to the top of a tower. Then, on a high-gravity day, you could lower the weight back down and use it to crank a generator, selling electric power back on the open market. You would have a kind of perpetual motion machine.

What you are buying from the electric company is a thing called energy, a term that has a specific technical meaning in physics. The fact that the law of gravity does does *not* seem to change over time tells us that we can't use a scheme like the one described above as a way to create energy out of nothing. In fact, experiments seem to show that no physical process can create or destroy energy, they can only transfer or transform it from one form into another. In other words, the total amount of energy in the universe can never change. A statement of this form is called a conservation law. Today, Newton's laws have been replaced by a set of conservation laws, including conservation of energy.

Writing conservation of energy symbolically, we have $E_1 + E_2 + \ldots = E_1' + E_2' + \ldots$, where the sum is over all the types of energy that are present, and the primed and unprimed letters distinguish the energies at some initial and final times. For more compact writing, we can use the

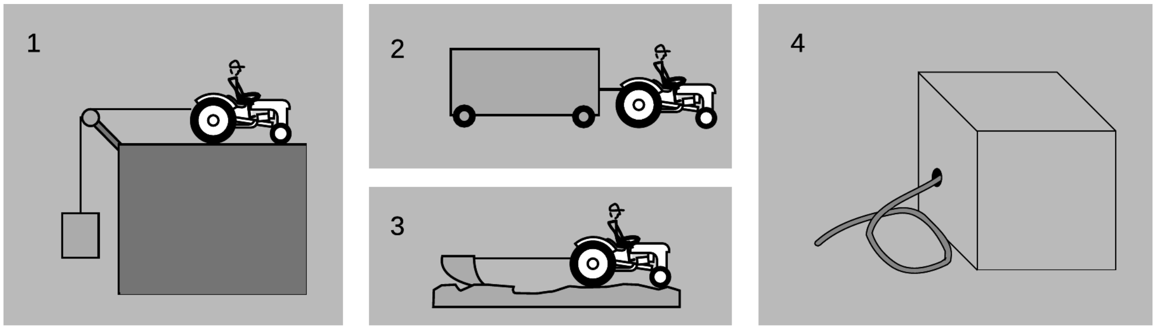


Figure 7.2: Work.

notation

$$\sum_k E_k = \sum_k E'_k, \tag{7.1}$$

where Σ, Greek uppercase sigma, stands for "sum," and k is an index taking on the values $1, 2, \ldots$.

Another conservation law is conservation of mass. Experiments by Lavoisier (1743-1794) showed that, for example, when wood was burned, the total mass of the smoke, hot gases, and charred wood was the same as the mass of the original wood. This view was modified in 1905 by Einstein's famous $E = mc^2$, which says that we can actually convert energy to mass and mass to energy. Therefore the separate laws of conservation of energy and mass are only approximations to a deeper, underlying conservation law that includes both quantities.

7.2 Work

Energy exists in various forms, such a the energy of sunlight, gravitational energy, and the energy of a moving object, called kinetic energy. Because it exists in so many forms, it is a tricky concept to define. By analogy, an amount of money can be expressed in terms of dollars, euros, or various other currencies, but modern governments no longer even attempt to define the value of their currencies in absolute terms such as ounces of gold. If we make radio contact with aliens someday, they will presumably not agree with us on how many units of energy there are in a liter of gasoline. We could, however, pick something arbitrary like a liter of gas as a standard of comparison.

Figure **??** shows a train of thought leading to a standard that turns out to be more convenient. In panel 1, the tractor raises the weight over the pulley. Gravitational energy is stored in the weight, and this energy could be released later by dropping or lowering the weight. In 2, the tractor accelerates the trailer, increasing its kinetic energy. In 3, the tractor pulls a plow. Energy is expended in frictional heating of the plow and the dirt, and in breaking dirt clods and lifting dirt up to the sides of the furrow. In all three examples, the energy of the gas in the tractor's tank is converted into some other form, and in all three examples there is a force F involved, and the tractor travels some distance d as it applies the force.

Now imagine a black box, panel 4, containing a gasoline-powered engine, which is designed to reel in a steel cable, exerting a force F. The box only communicates with the outside world via the hole through which its cable passes, and therefore the amount of energy transferred out through the cable can only depend on F and d. Since force and energy are both additive, this energy must be proportional to F, and since the energy transfer is additive as we reel in one section

of cable and then a further section, the energy must also be proportional to d. As an arbitrary standard, we pick the constant of proportionality to be 1, so that the energy transferred, notated W for *work*, is given by

$$W = Fd. \tag{7.2}$$

This equation implicitly defines the SI unit of energy to be $\mathrm{kg \cdot m^2/s^2}$, and we abbreviate this as one joule, $1\ \mathrm{J} = 1\ \mathrm{kg \cdot m^2/s^2}$.

In general, we define *work* as the transfer of energy by a macroscopic force, with a plus sign if energy is flowing from the object exerting the force to the object on which the force is exerted. (In examples such as heat conduction, the forces are forces that occur in the collisions between atoms, which are not measurable by macroscopic devices such as spring scales and force probes.) Equation (7.2) is a correct rule for computing work in the special case when the force is exerted at a single well-defined point of contact, that point moves along a line, the force and the motion are parallel, and the force is constant. The distance d is a signed quantity.

When the force and the motion are not parallel, we have the generalization

$$W = \mathbf{F} \cdot \Delta\mathbf{x}, \tag{7.3}$$

in which $\cdot$ is the vector dot product. When force and motion are along the same line, but the force is not constant,

$$W = \int F \, \mathrm{d}x. \tag{7.4}$$

Applying both of these generalizations at once gives

$$W = \int \mathbf{F} \cdot \mathrm{d}\mathbf{x}, \tag{7.5}$$

which is an example of the line integral from vector calculus.

The rate at which energy is transferred or transformed is the *power*,

$$P = \frac{\mathrm{d}E}{\mathrm{d}t}. \tag{7.6}$$

The units of power can be abbreviated as watts, $1\ \mathrm{W} = 1\ \mathrm{J/s}$. For the conditions under which $W = \int F \, \mathrm{d}x$ is valid, we can use the fundamental theorem of calculus to find $F = \mathrm{d}W/\mathrm{d}x$, and since $\mathrm{d}W/\mathrm{d}t = (\mathrm{d}K/\mathrm{d}x)(\mathrm{d}x/\mathrm{d}t)$, the power transmitted by the force is

$$P = Fv. \tag{7.7}$$

7.3 Kinetic energy

Having chosen mechanical work as an arbitrary standard for defining transfers of energy, we are led by Newton's laws to an expression for the energy that an object has because of its motion, called *kinetic energy*, K. One form of this work-kinetic energy theorem is as follows. Let a force act on a particle of mass m in one dimension. By the chain rule, we have $\mathrm{d}K/\mathrm{d}x = (\mathrm{d}K/\mathrm{d}v)(\mathrm{d}v/\mathrm{d}t)(\mathrm{d}t/\mathrm{d}x) = (\mathrm{d}K/\mathrm{d}v)a/v$. Applying $a = F/m$ and $\mathrm{d}K/\mathrm{d}x = F$ (work) gives $\mathrm{d}K/\mathrm{d}v = mv$. Integration of both sides with respect to v results in

$$K = \frac{1}{2}mv^2, \tag{7.8}$$

where the constant of integration can be taken to be zero. The factor of $1/2$ is ultimately a matter of convention; if we had wanted to avoid the $1/2$ in this equation, we could have, but we would have had to define work as $2Fd$.

When we heat an object, we are increasing the kinetic energy of the random motion of its molecules. The amount of energy required to heat one kilogram a substance by one degree is called its specific heat capacity. A useful figure is that the specific heat of water is $4.2\times10^3\ \mathrm{J/kg^\circ C}$.

7.4 Potential energy

Figure 7.3 shows someone lifting a heavy textbook at constant speed while a bug hitches a ride on top. The person's body has burned some calories, and although some of that energy went into body heat, an amount equal to Fd must

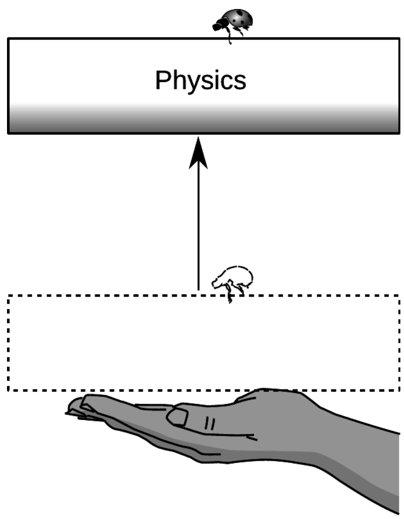


Figure 7.3: The book gains potential energy as it is raised.

have flowed into the book. But the bug may be excused for being skeptical about this. No measurement that the bug can do within its own immediate environment shows any changes in the properties of the book: there is no change in temperature, no vibration, nothing. Even if the bug looks at the walls of the room and is able to tell that it is rising, it finds that it is rising at constant speed, so there is no change in the book's kinetic energy. But if the person then takes her hand away and lets the book drop, the bug will have to admit that there is a spectacular and scary release of kinetic energy, which will later be transformed into sound and vibration when the book hits the floor.

If we are to salvage the law of conservation of energy, we are forced to invent a new type of energy, which depends on the height of the book in the earth's gravitational field. This is an energy of position, which is usually notated as PE or U. Any time two objects interact through a force exerted at a distance (gravity, magnetism, etc.), there is a corresponding position-energy, which is referred to as *potential energy.* The hand was giving gravitational potential energy to the book. Since the work done by the hand equals Fd, it follows that the potential energy must be given by

$$PE_{\mathrm{grav}} = mgy, \tag{7.9}$$

where an arbitary additive constant is implied because we have to choose a reference level at which to define $y = 0$. In the more general case where the external force such as gravity is not constant and can point in any direction, we have

$$\Delta PE = -\int_1^2 \mathbf{F} \cdot \mathrm{d}\mathbf{x}, \tag{7.10}$$

where 1 and 2 stand for the inital and final positions. This is a line integral, which is general may depend on the path the object takes. But for a certain class of forces, which includes, to a good approximation, the earth's gravitational force on an object, the result is *not* dependent on the path, and therefore the potential energy is well defined.

A common example is an elastic restoring force $F = -kx$ (Hooke's law), such as the force of a spring. Calculating $\Delta PE = -\int F\,\mathrm{d}x$, we find

$$PE = \frac{1}{2}kx^2, \tag{7.11}$$

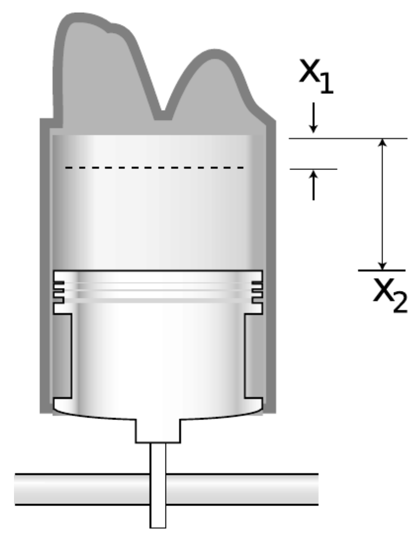
where the constant of integration is arbitrarily chosen to be zero. This is in fact a type of electrical potential energy, which varies as the lattice of atoms within the spring is distorted.

Problems

7-a1 "Big wall" climbing is a specialized type of rock climbing that involves going up tall cliffs such as the ones in Yosemite, usually with the climbers spending at least one night sleeping on a natural ledge or an artificial "portaledge." In this style of climbing, each pitch of the climb involves strenuously hauling up several heavy bags of gear — a fact that has caused these climbs to be referred to as "vertical ditch digging." (a) If an 80 kg haul bag has to be pulled up the full length of a 60 m rope, how much work is done? (b) Since it can be difficult to lift 80 kg, a 2:1 pulley is often used. The hauler then lifts the equivalent of 40 kg, but has to pull in 120 m of rope. How much work is done in this case? ✓

7-a2 An airplane flies in the positive direction along the x axis, through crosswinds that exert a force $\mathbf{F} = (a + bx)\hat{\mathbf{x}} + (c + dx)\hat{\mathbf{y}}$. Find the work done by the wind on the plane, and by the plane on the wind, in traveling from the origin to position x. ✓

7-a3 In the power stroke of a car's gasoline engine, the fuel-air mixture is ignited by the spark plug, explodes, and pushes the piston out. The exploding mixture's force on the piston head is greatest at the beginning of the explosion, and decreases as the mixture expands. It can be approximated by $F = a/x$, where x is the distance from the cylinder to the piston head, and a is a constant with units of $\mathrm{N \cdot m}$. (Actually $a/x^{1.4}$ would be more accurate, but the problem works out more nicely with a/x!) The piston begins its stroke at $x = x_1$, and ends at $x = x_2$. The 1965 Rambler had six cylinders, each with $a = 220\ \mathrm{N \cdot m}$, $x_1 = 1.2$ cm, and $x_2 = 10.2$ cm.
(a) Draw a neat, accurate graph of F vs x, on graph paper.
(b) From the area under the curve, derive the amount of work done in one stroke by one cylinder. ✓
(c) Assume the engine is running at 4800 r.p.m., so that during one minute, each of the six cylinders performs 2400 power strokes. (Power strokes only happen every other revolution.) Find the engine's power, in units of horsepower (1 hp=746 W). ✓
(d) The compression ratio of an engine is defined as x_2/x_1. Explain in words why the car's power would be exactly the same if x_1 and x_2 were, say, halved or tripled, maintaining the same compression ratio of 8.5. Explain why this would *not* quite be true with the more realistic force equation $F = a/x^{1.4}$.

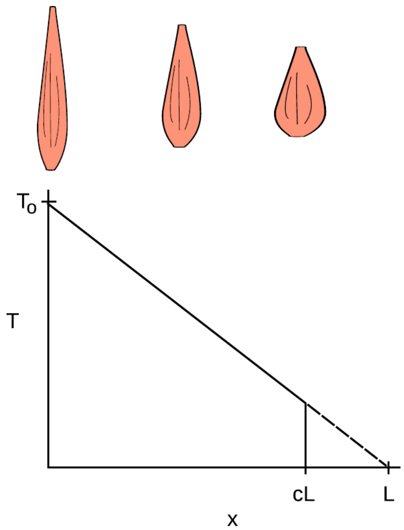


Problem 7-a3.

7-a4 (a) The crew of an 18th century warship is raising the anchor. The anchor has a mass of 5000 kg. The water is 30 m deep. The chain to which the anchor is attached has a mass per unit length of 150 kg/m. Before they start raising the anchor, what is the total weight of the anchor plus the portion of the chain hanging out of the ship? (Assume that the buoyancy of the anchor is negligible.)
(b) After they have raised the anchor by 1 m, what is the weight they are raising?
(c) Define $y = 0$ when the anchor is resting on the bottom, and $y = +30$ m when it has been raised up to the ship. Draw a graph of the force

the crew has to exert to raise the anchor and chain, as a function of y. (Assume that they are raising it slowly, so water resistance is negligible.) It will not be a constant! Now find the area under the graph, and determine the work done by the crew in raising the anchor, in joules. (d) Convert your answer from (c) into units of kcal. ✓

7-a5 The figure, redrawn from *Gray's Anatomy*, shows the tension of which a muscle is capable. The variable x is defined as the contraction of the muscle from its maximum length L, so that at $x = 0$ the muscle has length L, and at $x = L$ the muscle would theoretically have zero length. In reality, the muscle can only contract to $x = cL$, where c is less than 1. When the muscle is extended to its maximum length, at $x = 0$, it is capable of the greatest tension, T_o. As the muscle contracts, however, it becomes weaker. Gray suggests approximating this function as a linear decrease, which would theoretically extrapolate to zero at $x = L$. (a) Find the maximum work the muscle can do in one contraction, in terms of c, L, and T_o. ✓
(b) Show that your answer to part a has the right units.
(c) Show that your answer to part a has the right behavior when $c = 0$ and when $c = 1$.
(d) Gray also states that the absolute maximum tension T_o has been found to be approximately proportional to the muscle's cross-sectional area A (which is presumably measured at $x = 0$), with proportionality constant k. Approximating the muscle as a cylinder, show that your answer from part a can be reexpressed in terms of the volume, V, eliminating L and A. ✓
(e) Evaluate your result numerically for a biceps muscle with a volume of 200 cm^3, with $c = 0.8$ and $k = 100\ \text{N/cm}^2$ as estimated by Gray. ✓



Problem 7-a5.

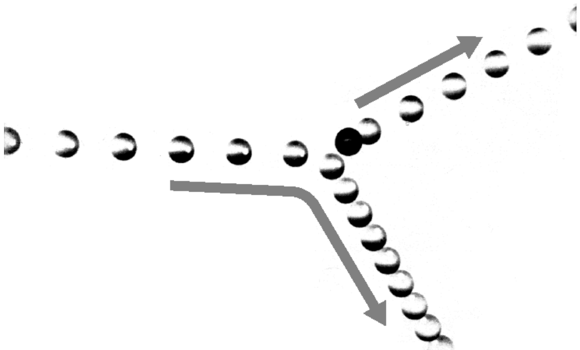
7-d1 Can kinetic energy ever be less than zero? Explain. [Based on a problem by Serway and Faughn.]

7-d2 A bullet flies through the air, passes through a paperback book, and then continues to fly through the air beyond the book. When is there a force? When is there energy?

7-d3 A 7.00 kg bowling ball moves at 3.00 m/s. How fast must a 2.45 g ping-pong ball move so that the two balls have the same kinetic energy? ✓

7-d4 You throw a ball straight up, and it lands with 2/3 the speed with which you threw it. What fraction of the initial kinetic energy was lost during the time the ball was in the air? ✓

7-d5 The multiflash photograph shows a collision between two pool balls. The ball that was initially at rest shows up as a dark image in its initial position, because its image was exposed several times before it was struck and began moving. By making *measurements* on the figure, determine *numerically* whether or not energy appears to have been conserved in the collision. What systematic effects would limit the accuracy of your test? [From an example in PSSC Physics.]



Problem 7-d5.

hw-colliding-balls

7-d6 You are driving your car, and you hit a brick wall head on, at full speed. The car has a mass of 1500 kg. The kinetic energy released is a measure of how much destruction will be done to the car and to your body. Calculate the energy released if you are traveling at (a) 40 mi/hr, and again (b) if you're going 80 mi/hr. What is counterintuitive about this, and what implication does this have for driving at high speeds? √

7-d7 Object A has a kinetic energy of 13.4 J. Object B has a mass that is greater by a factor of 3.77, but is moving more slowly by a factor of 2.34. What is object B's kinetic energy? [Based on a problem by Arnold Arons.]

7-d8 One theory about the destruction of the space shuttle Columbia in 2003 is that one of its wings had been damaged on liftoff by a chunk of foam insulation that fell off of one of its external fuel tanks. The New York Times reported on June 5, 2003, that NASA engineers had recreated the impact to see if it would damage a mock-up of the shuttle's wing. "Before last week's test, many engineers at NASA said they thought lightweight foam could not harm the seemingly tough composite panels, and privately predicted that the foam would bounce off harmlessly, like a Nerf ball." In fact, the 1.7-pound piece of foam, moving at 531 miles per hour, did serious damage. A member of the board investigating the disaster said this demonstrated that "people's intuitive sense of physics is sometimes way off." (a) Compute the kinetic energy of the foam, and (b) compare with the energy of a 170-pound boulder moving at 5.3 miles per hour (the speed it would have if you dropped it from about knee-level). √
(c) The boulder is a hundred times more massive, but its speed is a hundred times smaller, so what's counterintuitive about your results?

7-d9 A closed system can be a bad thing — for an astronaut sealed inside a space suit, getting rid of body heat can be difficult. Suppose a 60-kg astronaut is performing vigorous physical activity, expending 200 W of power. If none of the heat can escape from her space suit, how long will it take before her body temperature rises by 6°C (11°F), an amount sufficient to kill her? Assume that the amount of heat required to raise her body temperature by 1°C is the same as it would be for an equal mass of water. Express your answer in units of minutes. √

7-d10 Experiments show that the power consumed by a boat's engine is approximately proportional to the third power of its speed. (We assume that it is moving at constant speed.) (a) When a boat is crusing at constant speed, what type of energy transformation do you think is being performed? (b) If you upgrade to a motor with double the power, by what factor is your boat's crusing speed increased? [Based on a problem by Arnold Arons.]

7-g1 Estimate the kinetic energy of an Olympic sprinter.

7-g2 Estimate the kinetic energy of a buzzing fly's wing. (You may wish to review section 1.4 on order-of-magnitude estimates.)

7-g3 A blade of grass moves upward as it grows. Estimate its kinetic energy. (You may

wish to review section 1.4 on order-of-magnitude estimates.)

7-g4 All stars, including our sun, show variations in their light output to some degree. Some stars vary their brightness by a factor of two or even more, but our sun has remained relatively steady during the hundred years or so that accurate data have been collected. Nevertheless, it is possible that climate variations such as ice ages are related to long-term irregularities in the sun's light output. If the sun was to increase its light output even slightly, it could melt enough Antarctic ice to flood all the world's coastal cities. The total sunlight that falls on Antarctica amounts to about 1×10^{16} watts. In the absence of natural or human-caused climate change, this heat input to the poles is balanced by the loss of heat via winds, ocean currents, and emission of infrared light, so that there is no net melting or freezing of ice at the poles from year to year. Suppose that the sun changes its light output by some small percentage, but there is no change in the rate of heat loss by the polar caps. Estimate the percentage by which the sun's light output would have to increase in order to melt enough ice to raise the level of the oceans by 10 meters over a period of 10 years. (This would be enough to flood New York, London, and many other cities.) Melting 1 kg of ice requires 3×10^3 J.

7-j1 A ball rolls up a ramp, turns around, and comes back down. When does it have the greatest gravitational energy? The greatest kinetic energy? [Based on a problem by Serway and Faughn.]

7-j2 Can gravitational potential energy ever be negative? Note that the question refers to PE, not ΔPE, so that you must think about how the choice of a reference level comes into play. [Based on a problem by Serway and Faughn.]

7-j3 In each of the following situations, is the work being done positive, negative, or zero? (a) a bull paws the ground; (b) a fishing boat pulls a net through the water behind it; (c) the water resists the motion of the net through it; (d) you stand behind a pickup truck and lower a bale of hay from the truck's bed to the ground. Explain. [Based on a problem by Serway and Faughn.]

7-j4 (a) Suppose work is done in one-dimensional motion. What happens to the work if you reverse the direction of the positive coordinate axis? Base your answer directly on the definition of work. (b) Now answer the question based on the $W = Fd$ rule.

7-j5 Does it make sense to say that work is conserved?

7-j6 (a) You release a magnet on a tabletop near a big piece of iron, and the magnet leaps across the table to the iron. Does the magnetic energy increase, or decrease? Explain. (b) Suppose instead that you have two repelling magnets. You give them an initial push towards each other, so they decelerate while approaching each other. Does the magnetic energy increase, or decrease? Explain.

7-j7 Students are often tempted to think of potential energy and kinetic energy as if they were always related to each other, like yin and yang. To show this is incorrect, give examples of physical situations in which (a) PE is converted to another form of PE, and (b) KE is converted to another form of KE.

7-j8 Anya and Ivan lean over a balcony side by side. Anya throws a penny downward with an initial speed of 5 m/s. Ivan throws a penny upward with the same speed. Both pennies end up on the ground below. Compare their kinetic energies and velocities on impact.

7-j9 Decide whether the following statements regarding work and energy are true or false.

(a) The work done by a frictional force depends only on the initial and final points of the path of a particle.
(b) If a force is perpendicular to the direction of motion of an object, the force is not changing the kinetic energy of the object.
(c) The work done by a conservative force is zero.
(d) Doubling the amount of time a force is applied will double the work done on an object by the force.
(e) Since KE is always positive, the net work on a particle must be positive.

7-j10 When you buy a helium-filled balloon, the seller has to inflate it from a large metal cylinder of the compressed gas. The helium inside the cylinder has energy, as can be demonstrated for example by releasing a little of it into the air: you hear a hissing sound, and that sound energy must have come from somewhere. The total amount of energy in the cylinder is very large, and if the valve is inadvertently damaged or broken off, the cylinder can behave like a bomb or a rocket.

Suppose the company that puts the gas in the cylinders prepares cylinder A with half the normal amount of pure helium, and cylinder B with the normal amount. Cylinder B has twice as much energy, and yet the temperatures of both cylinders are the same. Explain, at the atomic level, what form of energy is involved, and why cylinder B has twice as much.

7-j11 Explain in terms of conservation of energy why sweating cools your body, even though the sweat is at the same temperature as your body. Describe the forms of energy involved in this energy transformation. Why don't you get the same cooling effect if you wipe the sweat off with a towel? Hint: The sweat is evaporating.

7-j12 A microwave oven works by twisting molecules one way and then the other, counterclockwise and then clockwise about their own centers, millions of times a second. If you put an ice cube or a stick of butter in a microwave, you'll observe that the solid doesn't heat very quickly, although eventually melting begins in one small spot. Once this spot forms, it grows rapidly, while the rest of the solid remains solid; it appears that a microwave oven heats a liquid much more rapidly than a solid. Explain why this should happen, based on the atomic-level description of heat, solids, and liquids.

Don't repeat the following common mistakes:
In a solid, the atoms are packed more tightly and have less space between them. Not true. Ice floats because it's *less* dense than water.
In a liquid, the atoms are moving much faster. No, the difference in average speed between ice at $-1°$C and water at $1°$C is only 0.4%.

7-j13 The figure above is from a classic 1920 physics textbook by Millikan and Gale. It represents a method for raising the water from the pond up to the water tower, at a higher level, without using a pump. Water is allowed into the drive pipe, and once it is flowing fast enough, it forces the valve at the bottom closed. Explain how this works in terms of conservation of mass and energy.

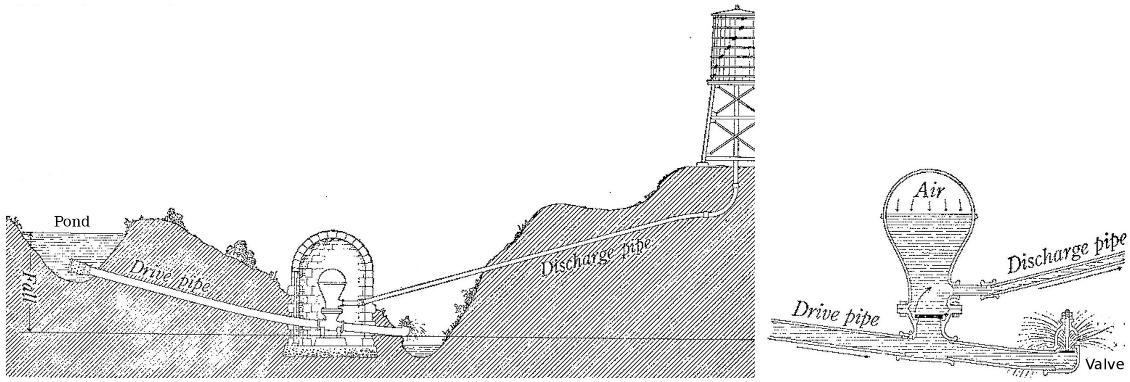
7-m1 A grasshopper with a mass of 110 mg falls from rest from a height of 310 cm. On the way down, it dissipates 1.1 mJ of heat due to air resistance. At what speed, in m/s, does it hit the ground?

▷ Solution, p. 202

7-m2 Suppose that the cost of energy in your city is 15 cents per kilowatt-hour. A cost-efficient light bulb uses energy at a rate of 25 W. How much does it cost to leave the light on for the entire month of January?

√

7-m3 Lisa times herself running up the stairs of her science building and finds that it takes her 23 s to reach the top floor. Her mass is 44 kg. If the vertical height reached is 11.0 m, what is minimum average power she would have to have produced during the climb (i.e., only taking into account the energy required to overcome gravity)?



Problem 7-j13.

√

7-m4 How long will it take a 3.92 kW motor, operating at full power, to lift a 1150 kg car to a height of 25.0 m? Assume frictional forces are negligible. (To make this more vivid for people in the US, 3.92 kW is 5.26 horsepower.)

√

7-m5 A roller coaster starts from rest and descends 35 meters in its initial drop and then rises 23 meters before going over a hill. A passenger at the top of the hill feels an apparent weight which is 2/3 of her normal weight. By using the fact that the energy loss due to friction must be greater than zero, find a bound on the radius of curvature of the first hill. Is this an upper bound, or a lower bound?

√

7-m6 A piece of paper of mass 4.5 g is dropped from a height 1.0 m above the ground. The paper dissipates 37 mJ of energy through frictional heating on its way down.
(a) How much kinetic energy does the paper have when it reaches the ground? √
(b) What is the speed of the paper when it hits the ground?

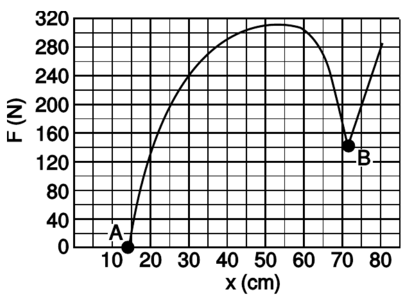
√

7-m7 Let E_b be the energy required to boil one kg of water. (a) Find an equation for the minimum height from which a bucket of water must be dropped if the energy released on impact is to vaporize it. Assume that all the heat goes into the water, not into the dirt it strikes, and ignore the relatively small amount of energy required to heat the water from room temperature to 100°C. [Numerical check, not for credit: Plugging in $E_b = 2.3$ MJ/kg should give a result of 230 km.] √
(b) Show that the units of your answer in part a come out right based on the units given for E_b.

7-m8 Most modern bow hunters in the U.S. use a fancy mechanical bow called a compound bow, which looks nothing like what most people imagine when they think of a bow and arrow. It has a system of pulleys designed to produce the force curve shown in the figure, where F is the force required to pull the string back, and x is the distance between the string and the center of the bow's body. It is not a linear Hooke's-law graph, as it would be for an old-fashioned bow. The big advantage of the design is that relatively little force is required to hold the bow stretched to point B on the graph. This is the force required from the hunter in order to hold the bow ready while waiting for a shot. Since it may be necessary to wait a long time, this force can't be too big. An old-fashioned bow, designed to require the same amount of force when fully drawn, would shoot arrows at much lower speeds, since its graph would be a straight line from A to B. For the graph shown in the figure (taken from realistic data), find the speed at which a

26 g arrow is released, assuming that 70% of the mechanical work done by the hand is actually transmitted to the arrow. (The other 30% is lost to frictional heating inside the bow and kinetic energy of the recoiling and vibrating bow.)

√



Problem 7-m8.

7-m9 The following table gives the amount of energy required in order to heat, melt, or boil a gram of water.

heat 1 g of ice by 1°C	2.05 J
melt 1 g of ice	333 J
heat 1 g of water by 1°C	4.19 J
boil 1 g of water	2500 J
heat 1 g of steam by 1°C	2.01 J

(a) How much energy is required in order to convert 1.00 g of ice at -20 °C into steam at 137 °C? √

(b) What is the minimum amount of hot water that could melt 1.00 g of ice?

√

7-m10 Lord Kelvin, a physicist, told the story of how he encountered James Joule when Joule was on his honeymoon. As he traveled, Joule would stop with his wife at various waterfalls, and measure the difference in temperature between the top of the waterfall and the still water at the bottom. (a) It would surprise most people to learn that the temperature increased. Why should there be any such effect, and why would Joule care? How would this relate to the energy concept, of which he was the principal inventor? (b) How much of a gain in temperature should there be between the top and bottom of a 50-meter waterfall? (c) What assumptions did you have to make in order to calculate your answer to part b? In reality, would the temperature change be more than or less than what you calculated? [Based on a problem by Arnold Arons.]

√

7-m11 Weiping lifts a rock with a weight of 1.0 N through a height of 1.0 m, and then lowers it back down to the starting point. Bubba pushes a table 1.0 m across the floor at constant speed, requiring a force of 1.0 N, and then pushes it back to where it started. (a) Compare the total work done by Weiping and Bubba. (b) Check that your answers to part a make sense, using the definition of work: work is the transfer of energy. In your answer, you'll need to discuss what specific type of energy is involved in each case.

7-p1 At a given temperature, the average kinetic energy per molecule is a fixed value, so for instance in air, the more massive oxygen molecules are moving more slowly on the average than the nitrogen molecules. The ratio of the masses of oxygen and nitrogen molecules is 16.00 to 14.01. Now suppose a vessel containing some air is surrounded by a vacuum, and the vessel has a tiny hole in it, which allows the air to slowly leak out. The molecules are bouncing around randomly, so a given molecule will have to "try" many times before it gets lucky enough to head out through the hole. Find the rate at which oxygen leaks divided by the rate at which nitrogen leaks. (Define this rate according to the fraction of the gas that leaks out in a given time, not the mass or number of molecules leaked per unit time.)

√

7-p2 In the earth's atmosphere, the molecules are constantly moving around. Because temperature is a measure of kinetic energy per molecule, the average kinetic energy of each type of molecule is the same, e.g., the average KE of the O_2 molecules is the same as the average KE of the N_2 molecules. (a) If the mass of an O_2

molecule is eight times greater than that of a He atom, what is the ratio of their average speeds? Which way is the ratio, i.e., which is typically moving faster? (b) Use your result from part a to explain why any helium occurring naturally in the atmosphere has long since escaped into outer space, never to return. (Helium is obtained commercially by extracting it from rocks.) You may want to do problem 11-s1 first, for insight.
✓

7-p3 Two speedboats are identical, but one has more people aboard than the other. Although the total masses of the two boats are unequal, suppose that they happen to have the same kinetic energy. In a boat, as in a car, it's important to be able to stop in time to avoid hitting things. (a) If the frictional force from the water is the same in both cases, how will the boats' stopping distances compare? Explain. (b) Compare the times required for the boats to stop.

7-p4 A car starts from rest at $t = 0$, and starts speeding up with constant acceleration. (a) Find the car's kinetic energy in terms of its mass, m, acceleration, a, and the time, t. (b) Your answer in the previous part also equals the amount of work, W, done from $t = 0$ until time t. Take the derivative of the previous expression to find the power expended by the car at time t. (c) Suppose two cars with the same mass both start from rest at the same time, but one has twice as much acceleration as the other. At any moment, how many times more power is being dissipated by the more quickly accelerating car? (The answer is not 2.)
✓

7-p5 While in your car on the freeway, you're travelling at a constant speed of 55 miles/hour, requiring a power output of 50 horsepower from the engine. Almost all of the energy provided by the engine is used to fight air resistance, which is proportional in magnitude to the square of the speed of the car. If you step on the gas pedal all the way and increase the power output to 100 horsepower, what final speed will you reach? Note that this problem can be done without any conversions or knowledge of US units.
✓

7-s1 A soccer ball of mass m is moving at speed v when you kick it in the same direction it is moving. You kick it with constant force F, and you want to triple the ball's speed. Over what distance must your foot be in contact with the ball?
✓

7-s2 A laptop of mass m and a desktop computer of mass $3m$ are both dropped from the top of a building. The laptop has kinetic energy K when it reaches the ground.
(a) Find the kinetic energy of the desktop machine on impact, in terms of K, m, or both. ✓
(b) Find its speed in terms of the same variables.
✓

7-s3 A girl picks up a stone of mass m from the ground and throws it at speed v, releasing the stone from a height h above the ground. If the maximum power output of the girl is P, how many stones could she throw in a time T?

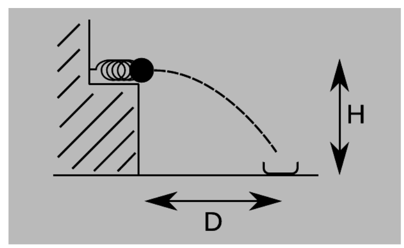
7-s4 A car accelerates from rest. At low speeds, its acceleration is limited by static friction, so that if we press too hard on the gas, we will "burn rubber" (or, for many newer cars, a computerized traction-control system will override the gas pedal). At higher speeds, the limit on acceleration comes from the power of the engine, which puts a limit on how fast kinetic energy can be developed.
(a) Show that if a force F is applied to an object moving at speed v, the power required is given by $P = vF$.
(b) Find the speed v at which we cross over from the first regime described above to the second. At speeds higher than this, the engine does not have enough power to burn rubber. Express your result in terms of the car's power P, its mass m, the coefficient of static friction μ_s, and g. ✓
(c) Show that your answer to part b has units that make sense.
(d) Show that the dependence of your answer on each of the four variables makes sense physically.

(e) The 2010 Maserati Gran Turismo Convertible has a maximum power of 3.23×10^5 W (433 horsepower) and a mass (including a 50-kg driver) of 2.03×10^3 kg. (This power is the maximum the engine can supply at its optimum frequency of 7600 r.p.m. Presumably the automatic transmission is designed so a gear is available in which the engine will be running at very nearly this frequency when the car is moving at v.) Rubber on asphalt has $\mu_s \approx 0.9$. Find v for this car. Answer: 18 m/s, or about 40 miles per hour.
(f) Our analysis has neglected air friction, which can probably be approximated as a force proportional to v^2. The existence of this force is the reason that the car has a maximum speed, which is 176 miles per hour. To get a feeling for how good an approximation it is to ignore air friction, find what fraction of the engine's maximum power is being used to overcome air resistance when the car is moving at the speed v found in part e. Answer: 1%

7-s5 A piece of paper of mass m is dropped from a height H above the ground. Because of air resistance, the paper lands with only 1/5th the speed that it would have landed with had there been no air resistance.
(a) What is the work on the paper due to gravity? ✓
(b) What is the work on the paper due to the drag force?
✓

7-s6 A block of mass m is at the top of a ramp of length L inclined at angle θ with respect to the horizontal. The block slides down the ramp, and the coefficient of friction between the two surfaces is μ_k. Parts a-c are about finding the speed of the block when it reaches the bottom of the ramp.
(a) Based on units, infer as much as possible about the form of the answer.
(b) Find the speed. ✓
(c) Check that the dependence of the result on the variables make sense. Under what conditions is the result unphysical?
(d) The speed of the block at the bottom of the ramp is only half of what it would have been without friction. Knowing this, what is the coefficient of friction μ_k in terms of the other given quantities?
✓

7-s7 Some kids are playing a game where they shoot a ball of mass m off a spring into a cup that is a distance D away from the base of the table (see figure). The ball starts at a height H, and the spring has spring constant k. The goal of the problem is to find the distance you should compress the spring so that the ball lands in the cup.
(a) Infer as much as possible about the form of the result based on units.
(b) Find the result.
✓



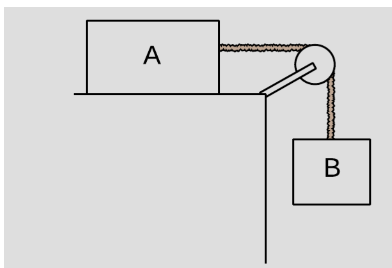
Problem 7-s7.

7-s8 A person on a bicycle is to coast down a ramp of height h and then pass through a circular loop of radius r. What is the smallest value of h for which the cyclist will complete the loop without falling? (Ignore the kinetic energy of the spinning wheels.) ✓

7-s9 Suppose that the cyclist in problem 7-s8 wants to have a little extra security, passing through the top of the loop at twice the minimum speed that would have theoretically been required.
(a) What is this speed? ✓
(b) Find the height of the ramp that is needed.
✓

7-s10 Consider the system shown in the figure. Block A has mass M_A and block B has mass

M_B. There is a coefficient of kinetic friction μ_k between block A and the table. The system is released from rest and block B drops a distance D.
(a) What is the work done by gravity on block B? ✓
(b) What is the tension in the string? ✓
(c) What is the work done on block B by the string? ✓
(d) By adding your results in parts a and c, find the speed of block B. ✓
(e) Show that the sum of the works done on block A equals the change in KE of block A.

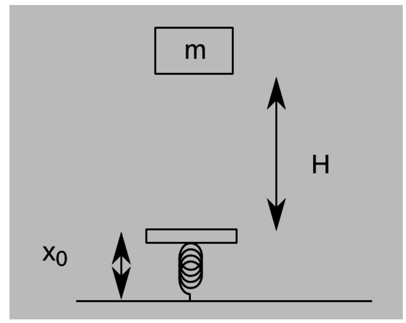


Problem 7-s10.

7-s11 (a) A circular hoop of mass m and radius r spins like a wheel while its center remains at rest. Its period (time required for one revolution) is T. Show that its kinetic energy equals $2\pi^2 mr^2/T^2$.
(b) If such a hoop rolls with its center moving at velocity v, its kinetic energy equals $(1/2)mv^2$, plus the amount of kinetic energy found in the first part of this problem. Show that a hoop rolls down an inclined plane with half the acceleration that a frictionless sliding block would have.

7-s12 As shown in the figure, a box of mass m is dropped from a height H above a spring. The platform has negligible mass. The spring has spring constant k.
(a) By what amount is the spring compressed? Your answer requires the quadratic equation; to choose the correct root, note that x needs to be positive. ✓
(b) What is the speed of the box when it is first in contact with the platform? ✓
(c) What is the maximum speed of the box? *Hint: the box speeds up until the force from the spring equals the gravitational force.*

✓ ⋆



Problem 7-s12.

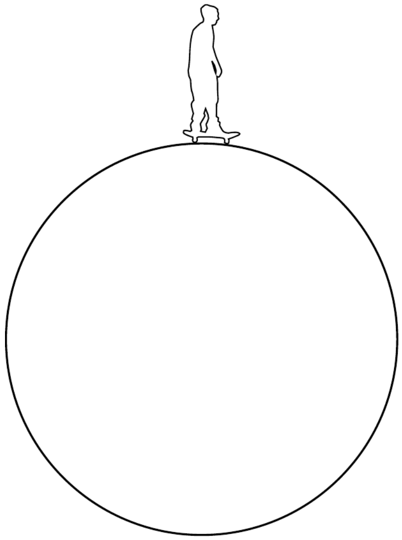
7-s13 A skateboarder starts at rest nearly at the top of a giant cylinder, and begins rolling down its side. (If he started exactly at rest and exactly at the top, he would never get going!) Show that his board loses contact with the pipe after he has dropped by a height equal to one third the radius of the pipe.

⋆

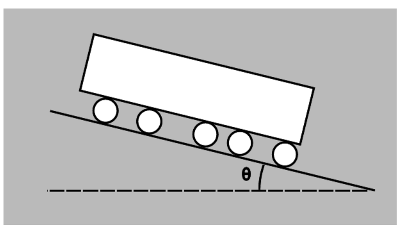
7-s14 The figure shows a slab of mass M rolling freely down an inclined plane inclined at an angle θ to the horizontal. The slab is on top of a set of rollers, each of radius r, that roll without slipping at their top and bottom surfaces. The rollers may for example be cylinders, or spheres such as ball bearings. Each roller's center of mass coincides with its geometrical center. The sum of the masses of the rollers is m, and the sum of their moments of inertia (each about its own center) is I. Find the acceleration of the slab, and verify that your expression has the correct behavior in interesting limiting cases.

✓ ⋆

7-v1 The magnitude of the force between two magnets separated by a distance r can be approximated as kr^{-3} for large values of r. The



Problem 7-s13.



Problem 7-s14.

constant k depends on the strengths of the magnets and the relative orientations of their north and south poles. Two magnets are released on a slippery surface at an initial distance r_i, and begin sliding towards each other. What will be the total kinetic energy of the two magnets when they reach a final distance r_f? (Ignore friction.) ✓

7-v2 An object's potential energy is described by the function $U(x) = -\alpha x^2 + \beta x^4$, where α and β are positive constants.
(a) For what positive value of x is the force on the object equal to zero? ✓
(b) What is the force on the particle when $x = 2\sqrt{\alpha/\beta}$? ✓

7-v3 The potential energy of a particle moving in a certain one-dimensional region of space is

$$U(x) = (1.00\ \mathrm{J/m^3})x^3 - (7.00\ \mathrm{J/m^2})x^2 + (10.0\ \mathrm{J/m})x.$$

(a) Determine the force $F(x)$ acting on the particle as a function of position.
(b) Is the force you found in part a conservative, or non-conservative? Explain.
(c) Let "$\mathcal{R}$" refer to the region $x = -1.00$ m to $x = +6.00$ m. Draw $U(x)$ on $\mathcal{R}$ (label your axes). On your plot, label all points of stable and unstable equilibrium on $\mathcal{R}$, and find their locations.
(d) What is the maximum force (in magnitude) experienced by a particle on $\mathcal{R}$?
(e) The particle has mass 1.00 kg and is released from rest at $x = 2.00$ m. Describe the subsequent motion. What is the maximum KE that the particle achieves?

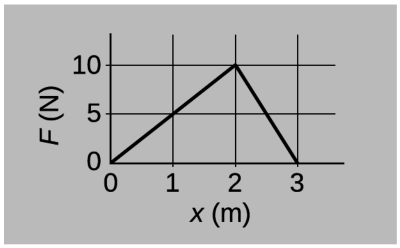
7-v4 The potential energy of a particle moving in a certain one-dimensional region of space is

$$U(x) = (1.00\ \mathrm{J/m^4})x^4 - (4.00\ \mathrm{J/m^2})x^2 + (1.00\ \mathrm{J/m})x.$$

You might want to plot the function $U(x)$ using a graphing calculator or an online utility such as desmos.com. An object of mass 2.00 kg is released from rest at $x = 2.00$ m.
(a) At what position does the object have maximum speed? ✓
(b) What is the maximum speed of the object? ✓
(c) What is the velocity of the object when it reaches the unstable equilibrium point? ✓
(d) What is the lowest value of x that the object gets to? ✓

7-v5 A banana starts at rest and is subject to the force shown. This force is the only force acting on the banana.

(a) What is the work done on the banana as it moves from $x = 0$ m to 2 m? ✓
(b) From 2 m to 3 m? ✓
(c) If this force acts on the banana for 0.5 s, what is the average power delivered to the banana? ✓



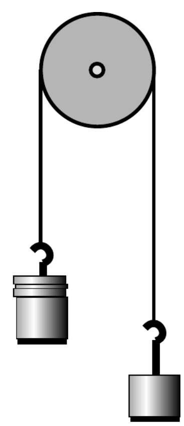
Problem 7-v5.

7-v6 An object of mass m is moving with speed v in the $+x$-direction when, starting at $x = 0$, it is subjected to the position-dependent force $F(x) = -kx^2$, where k is a positive constant. What is the maximum x-coordinate the object will reach? ✓

7-v7 In 1935, Yukawa proposed an early theory of the force that held the neutrons and protons together in the nucleus. His equation for the potential energy of two such particles, at a center-to-center distance r, was $PE(r) = gr^{-1}e^{-r/a}$, where g parametrizes the strength of the interaction, e is the base of natural logarithms, and a is about 10^{-15} m. Find the force between two nucleons that would be consistent with this equation for the potential energy. ✓

7-v8 A rail gun is a device like a train on a track, with the train propelled by a powerful electrical pulse. Very high speeds have been demonstrated in test models, and rail guns have been proposed as an alternative to rockets for sending into outer space any object that would be strong enough to survive the extreme accelerations. Suppose that the rail gun capsule is launched straight up, and that the force of air friction acting on it is given by $F = be^{-cx}$, where x is the altitude, b and c are constants, and e is the base of natural logarithms. The exponential decay occurs because the atmosphere gets thinner with increasing altitude. (In reality, the force would probably drop off even faster than an exponential, because the capsule would be slowing down somewhat.) Find the amount of kinetic energy lost by the capsule due to air friction between when it is launched and when it is completely beyond the atmosphere. (Gravity is negligible, since the air friction force is much greater than the gravitational force.) ✓

7-v9 The figure shows two unequal masses, M and m, connected by a string running over a pulley. This system was analyzed previously in problem 5-p2 on p. 63, using Newton's laws.
(a) Analyze the system using conservation of energy instead. Find the speed the weights gain after being released from rest and traveling a distance h. ✓
(b) Use your result from part a to find the acceleration, reproducing the result of the earlier problem. ✓



Problem 7-v9.

7-v10 A mass moving in one dimension is attached to a horizontal spring. It slides on the surface below it, with equal coefficients of static and

kinetic friction, $\mu_k = \mu_s$. The equilibrium position is $x = 0$. If the mass is pulled to some initial position and released from rest, it will complete some number of oscillations before friction brings it to a stop. When released from $x = a$ ($a > 0$), it completes exactly 1/4 of an oscillation, i.e., it stops precisely at $x = 0$. Similarly, define $b > 0$ as the greatest x from which it could be released and comlete 1/2 of an oscillation, stopping on the far side and not coming back toward equilibrium. Find b/a. Hint: To keep the algebra simple, set every fixed parameter of the system equal to 1.

√

7-v11 In 2003, physicist and philosopher John Norton came up with the following apparent paradox, in which Newton's laws, which appear deterministic, can produce nondeterministic results. Suppose that a bead moves frictionlessly on a curved wire under the influence of gravity. The shape of the wire is defined by the function $y(x)$, which passes through the origin, and the bead is released from rest at the origin. For convenience of notation, choose units such that $g = 1$, and define $\dot{y} = \mathrm{d}y/\,\mathrm{d}t$ and $y' = \mathrm{d}y/\,\mathrm{d}x$.

(a) Show that the equation of motion is

$$y = -\frac{1}{2}\dot{y}^2\left(1 + y'^{-2}\right).$$

(b) To simplify the calculations, assume from now on that $y' \ll 1$. Find a shape for the wire such that $x = t^4$ is a solution. (Ignore units.) √

(c) Show that not just the motion assumed in part b, but any motion of the following form is a solution:

$$x = \begin{cases} 0 & \text{if } t \leq t_0 \\ (t - t_0)^4 & \text{if } t \geq t_0 \end{cases}$$

This is remarkable because there is no physical principle that determines t_0, so if we place the bead at rest at the origin, there is no way to predict when it will start moving.

⋆

8 Conservation of momentum

This is not a textbook. It's a book of problems meant to be used along with a textbook. Although each chapter of this book starts with a brief summary of the relevant physics, that summary is not meant to be enough to allow the reader to actually learn the subject from scratch. The purpose of the summary is to show what material is needed in order to do the problems, and to show what terminology and notation are being used.

8.1 Momentum: a conserved vector

Consider the hockey puck in figure 8.1. If we release it at rest, we expect it to remain at rest. If it did start moving all by itself, that would be strange: it would have to pick some direction in which to move, and why would it have such a deep desire to visit the region of space on one side rather than the other? Such behavior, which is not actually observed, would suggest that the laws of physics differed between one region of space and another.

Figure 8.1: A hockey puck is released at rest. Will it start moving in some direction?

The laws of physics are in fact observed to be the same everywhere, and this symmetry leads to a conservation law, conservation of *momentum*. The momentum of a material object, notated $\mathbf{p}$ for obscure reasons, is given by the product of its mass and its momentum,

$$\mathbf{p} = m\mathbf{v}. \tag{8.1}$$

From the definition, we see that momentum is a vector. That's important because up until now, the only conserved quantities we'd encountered were mass and energy, which are both scalars. Clearly the laws of physics would be incomplete if we never had a law of physics that related to the fact that the universe has three dimensions of space. For example, it wouldn't violate conservation of mass or energy if an object was moving in a certain direction and then suddenly changed its direction of motion, while maintaining the same speed.

If we differentiating the equation for momentum with respect to time and apply Newton's second law, we obtain

$$\mathbf{F}_{\text{total}} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}. \tag{8.2}$$

We can also see from this equation that in the special case of a system of particles, conservation of momentum is closely related to Newton's third law.

8.2 Collisions

A collision is an interaction between particles in which the particles interact over some period of time and then stop interacting. It is assumed that external forces are negligible. Often the result of a collision can be uniquely predicted by simultaneously imposing conservation of energy and conservation of momentum,

$$\sum E_j = \sum E'_j \tag{8.3}$$

$$\sum \mathbf{p}_k = \sum \mathbf{p}'_k, \tag{8.4}$$

where j runs over the types of energy and k over the particles.

Some collisions are highly elastic, meaning that little or no kinetic energy is transformed into other forms, such as heat or sound. Other collisions, for example car crashes, are highly inelastic. Unless we have some reason to believe that the collision is elastic, we cannot assume that the total *kinetic* energy is conserved.

8.3 The center of mass

Figure 4.4 on p. 44 showed two ice skaters, initially at rest, pushing off from each other in opposite directions. If their masses are equal, then the average of their positions, $(x_1 + x_2)/2$, remains at rest. Generalizing this to more than one dimension, more than two particles, and possibly unequal masses, we define the *center of mass* of a system of particles to be

$$\mathbf{x}_{\text{cm}} = \frac{\sum m_i \mathbf{x}_i}{\sum m_i}, \tag{8.5}$$

which is a weighted average of all the position vectors $\mathbf{x}_i$. The velocity $\mathbf{v}_{\text{cm}}$ with which this point moves is related to the total momentum of the system by

$$\mathbf{p}_{\text{total}} = m_{\text{total}} \mathbf{v}_{\text{cm}}. \tag{8.6}$$

If no external force acts on the system, it follows that $\mathbf{v}_{\text{cm}}$ is constant. Often problems can be simplified by adopting the center of mass frame of reference, in which $\mathbf{v}_{\text{cm}} = 0$.

Problems

8-a1 When the contents of a refrigerator cool down, the changed molecular speeds imply changes in both momentum and energy. Why, then, does a fridge transfer *power* through its radiator coils, but not *force*?

▷ Solution, p. 202

8-a2 A firework shoots up into the air, and just before it explodes it has a certain momentum and kinetic energy. What can you say about the momenta and kinetic energies of the pieces immediately after the explosion? [Based on a problem from PSSC Physics.]

▷ Solution, p. 202

8-a3 Two people in a rowboat wish to move around without causing the boat to move. What should be true about their total momentum? Explain.

8-a4 Two blobs of putty collide head-on and stick. The collision is completely symmetric: the blobs are of equal mass, and they collide at equal speeds. What becomes of the energy the blobs had before the collision? The momentum?

8-d1 Derive a formula expressing the kinetic energy of an object in terms of its momentum and mass.
√

8-d2 Show that for a body made up of many *equal* masses, the equation for the center of mass becomes a simple average of all the positions of the masses.

8-d3 Objects of mass m and $4m$ are dropped from the top of a building (both starting from rest). When it hits the ground, the object of mass m has momentum p. What is the momentum of the heavier object when it hits the ground?

8-d4 The force acting on an object is $F = Ae^{-t/\tau}$, where A and τ are positive constants. The object is at rest at time $t = 0$.
(a) What is the momentum of the object at time $t = \tau$? √
(b) What is the final momentum of the object?
√

8-d5 The force acting on an object is $F = At^2$. The object is at rest at time $t = 0$. What is its momentum at $t = T$?
√

8-g1 Decide whether the following statements about one-dimensional motion are true or false:
(a) The momentum transferred to an object is equal to the final momentum of the object.
(b) The momentum delivered to an object by a force F is equal to the average force on the object multiplied by the time over which the force acts on the object.
(c) Momentum transfer has the same dimensions as force (SI units of newtons).
(d) The area underneath a momentum-vs-time graph gives the average force delivered to an object.

8-g2 Can the result of a collision always be determined by the condition that both energy and momentum are conserved? If your answer is no, give a counterexample.

8-g3 The big difference between the equations for momentum and kinetic energy is that one is proportional to v and one to v^2. Both, however, are proportional to m. Suppose someone tells you that there's a third quantity, funkosity, defined as $f = m^2v$, and that funkosity is conserved. How do you know your leg is being pulled?

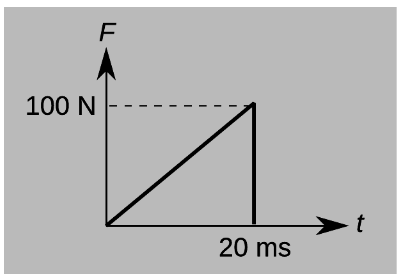
▷ Solution, p. 202 ⋆

8-j1 A mass m moving at velocity v collides with a stationary target having the same mass m. Find the maximum amount of energy that can be released as heat and sound.
√

8-j2 A bullet leaves the barrel of a gun with a kinetic energy of 90 J. The gun barrel is 50 cm long. The gun has a mass of 4 kg, the bullet 10 g.

(a) Find the bullet's final velocity. ✓
(b) Find the bullet's final momentum. ✓
(c) Find the momentum of the recoiling gun.
(d) Find the kinetic energy of the recoiling gun, and explain why the recoiling gun does not kill the shooter.
✓

8-j3 The figure shows the force acting on a 58.5 g tennis ball as a function of time.
(a) What is the momentum transferred to the tennis ball? ✓
(b) What is the final speed of the tennis ball if it is initially at rest? ✓
(c) What is the final speed of the tennis ball if its initial velocity is -25m/s?
✓



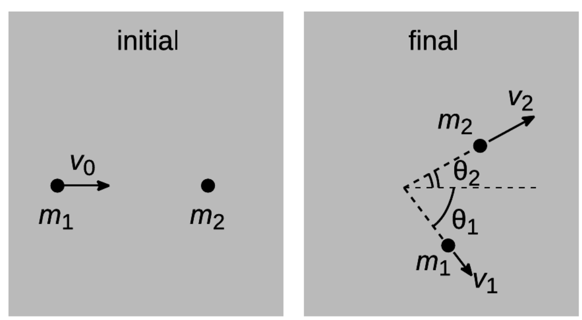
Problem 8-j3.

8-j4 A learjet traveling due east at 300 mi/hr collides with a jumbo jet which was heading southwest at 150 mi/hr. The jumbo jet's mass is five times greater than that of the learjet. When they collide, the learjet sticks into the fuselage of the jumbo jet, and they fall to earth together. Their engines stop functioning immediately after the collision. On a map, what will be the direction from the location of the collision to the place where the wreckage hits the ground? (Give an angle.)
✓

8-j5 A 1000 kg car approaches an intersection traveling north at 20.0 m/s. A 1200 kg car approaches the same intersection traveling east at 22.0 m/s. The two cars collide at the intersection and lock together. The drivers probably wish it was all over now, but they're still moving. What is the velocity of the cars immediately after the collision?
✓

8-j6 Two equal masses travel at equal speeds and collide in a perfectly inelastic collision. The final velocity of the two masses is 1/3 the initial speed. What was the angle between the velocity vectors of the two masses when they collided? (Give an exact expression, not a decimal approximation.)
✓

8-j7 A ball of mass m_1 is moving to the right at speed v_0 when it collides with a ball of mass m_2 initially at rest. After the collision, m_1 loses 75% of its initial kinetic energy and has a velocity at an angle $\theta_1 = 60°$ below the horizontal, as shown.
(a) What is the speed of ball 1 after the collision?
✓
(b) What are the x and y components of the momentum of ball 1 after the collision? ✓
(c) What are the x and y components of the momentum of ball 2 after the collision? ✓
(d) What fraction of the initial kinetic energy does ball 2 have after the collision? ✓
(e) Is this collision elastic, or inelastic? Explain.

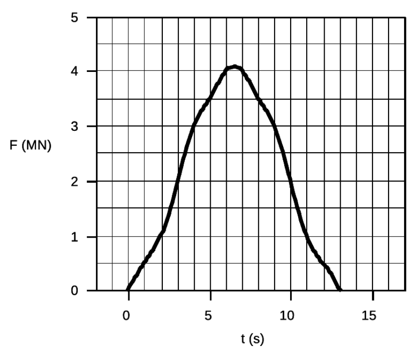


Problem 8-j7.

hw-collision-2d-given-one-angle
8-j8 The graph shows the force, in meganewtons, exerted by a rocket engine on the rocket as a function of time. If the rocket's mass is 4000 kg, at what speed is the rocket moving when the

engine stops firing? Assume it goes straight up, and neglect the force of gravity, which is much less than a meganewton.

✓



Problem 8-j8.

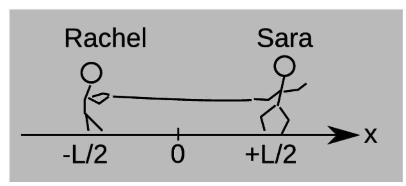
hw-rocket

8-j9 Cosmic rays are particles from outer space, mostly protons and atomic nuclei, that are continually bombarding the earth. Most of them, although they are moving extremely fast, have no discernible effect even if they hit your body, because their masses are so small. Their energies vary, however, and a very small minority of them have extremely large energies. In some cases the energy is as much as several Joules, which is comparable to the KE of a well thrown rock! If you are in a plane at a high altitude and are so incredibly unlucky as to be hit by one of these rare ultra-high-energy cosmic rays, what would you notice, the momentum imparted to your body, the energy dissipated in your body as heat, or both? Base your conclusions on numerical estimates, not just random speculation. (At these high speeds, one should really take into account the deviations from Newtonian physics described by Einstein's special theory of relativity. Don't worry about that, though.)

8-j10 A 10-kg bowling ball moving at 2.0 m/s hits a 1.0-kg bowling pin, which is initially at rest. The other pins are all gone already, and the collision is head-on, so that the motion is one-dimensional. Assume that negligible amounts of heat and sound are produced. Find the velocity of the pin immediately after the collision.

8-m1 A student of mass M is traveling on his skateboard of mass m. They are both moving at speed v, when suddenly the student kicks the board back so that it is immediately at rest relative to the ground. How fast is the student moving after kicking back the skateboard?

8-m2 Rachel and Sara are playing a game of tug-of-war on frictionless ice. They are separated by a distance L, and the coordinate system is given in the figure (with Sara at $x = +L/2$ and Rachel at $x = -L/2$). Rachel has a mass M_R, Sara has a mass M_S, and $M_R > M_S$. Where will they meet?

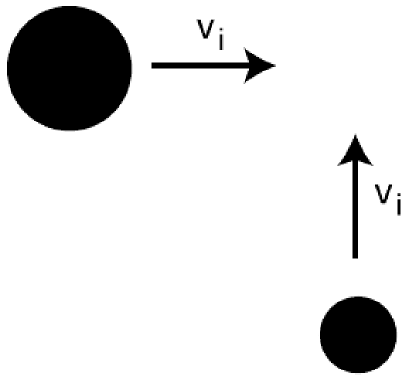


Problem 8-m2.

8-m3 The figure shows a view from above of a collision about to happen between two air hockey pucks sliding without friction. They have the same speed, v_i, before the collision, but the big puck is 2.3 times more massive than the small one. Their sides have sticky stuff on them, so when they collide, they will stick together. At what angle will they emerge from the collision? In addition to giving a numerical answer, please indicate by drawing on the figure how your angle is defined.

▷ Solution, p. 202

8-m4 The moon doesn't really just orbit the Earth. By Newton's third law, the moon's gravitational force on the earth is the same as the earth's force on the moon, and the earth must respond to the moon's force by accelerating. If we consider the earth and moon in isolation and ignore outside forces, then Newton's first law says



Problem 8-m3.

their common center of mass doesn't accelerate, i.e., the earth wobbles around the center of mass of the earth-moon system once per month, and the moon also orbits around this point. The moon's mass is 81 times smaller than the earth's. Compare the kinetic energies of the earth and moon.

8-m5 A very massive object with velocity v collides head-on with an object at rest whose mass is very small. No kinetic energy is converted into other forms. Prove that the low-mass object recoils with velocity $2v$. [Hint: Use the center-of-mass frame of reference.]

8-m6 An ice puck of mass m, traveling with speed v, hits another identical ice puck that is sitting at rest. The collision is 1-dimensional.
(a) If the collision is perfectly elastic, what is the final speed of the puck that was initially at rest? ✓
(b) If the collision is perfectly inelastic, what is the final speed of the two pucks after the collision? ✓
(c) If the collision is perfectly inelastic, what fraction of the total energy was lost during the collision? ✓
(d) If one-fourth of the initial kinetic energy was lost during the collision, what is the final speed of the puck that was initially at rest? ✓

8-m7 A bullet of mass m strikes a block of mass M which is hanging by a string of length L from the ceiling. It is observed that, after the sticky collision, the maximum angle that the string makes with the vertical is θ. This setup is called a ballistic pendulum, and it can be used to measure the speed of the bullet.
(a) What vertical height does the block reach? ✓
(b) What was the speed of the block just after the collision? ✓
(c) What was the speed of the bullet just before it struck the block? ✓

8-m8 A car of mass M and a truck of mass $2M$ collide head-on with equal speeds v, and the collision is perfectly inelastic, i.e., the maximum possible amount of kinetic energy is transformed into heat and sound, consistent with conservation of momentum.
(a) What is the magnitude of the change in momentum of the car? ✓
(b) What is the magnitude of the change in momentum of the truck? ✓
(c) What is the final speed of the two vehicles? ✓
(d) What fraction of the initial kinetic energy was lost as a result of the collision? ✓

8-m9 A 5.00 kg firework is launched straight up into the air. When it reaches its maximum height of $H = 140$ m, it explodes into two fragments that fly off in horizontal directions. The explosion is very quick, and only lasts 15 ms. One of the two fragments (fragment A, with mass $M_A = 2.00$ kg) lands 290 meters away from the initial launch position.
(a) Find the speed of fragment A just after the explosion. ✓
(b) By using conservation of momentum and your answer from part a, find the speed of the other fragment (call this fragment B) just after the explosion. ✓
(c) Calculate the magnitude of the momentum transferred to fragment A due to the explosion.

✓

(d) Calculate the magnitude of the impulse delivered by gravity to fragment A over the course of the explosion. ✓

(e) How far away from the initial launch position does fragment B land? *Hint: the center of mass of the two fragments lands at the location where the firework was initially launched.*

✓

8-m10 An object of mass m, moving at velocity u, undergoes a one-dimensional elastic collision with a mass km that is initially at rest. Let the positive direction be in the direction of the initial motion, so that $u > 0$. (a) What is the final velocity of mass m? ✓

(b) What is the final velocity of the mass km?

✓ ⋆

8-m11 Two blocks, each of mass M, are connected by a thread and moving with speed v_0. Between them is also a spring of spring constant k, and it is compressed a distance x (so that the tension in the thread is kx). Suddenly, the thread breaks, and the spring relaxes to its equilibrium length. Find the speed of the block that is pushed forward by the spring.

⋆

8-m12 Suppose a system consisting of pointlike particles has a total kinetic energy K_{cm} measured in the center-of-mass frame of reference. Since they are pointlike, they cannot have any energy due to internal motion.

(a) Prove that in a different frame of reference, moving with velocity $\mathbf{u}$ relative to the center-of-mass frame, the total kinetic energy equals $K_{cm} + M|\mathbf{u}|^2/2$, where M is the total mass. [Hint: You can save yourself a lot of writing if you express the total kinetic energy using the dot product.]

(b) Use this to prove that if energy is conserved in one frame of reference, then it is conserved in every frame of reference. The total energy equals the total kinetic energy plus the sum of the potential energies due to the particles' interactions with each other, which we assume depends only on the distance between particles.

8-m13 A flexible rope of mass m and length L slides without friction over the edge of a table. Let x be the length of the rope that is hanging over the edge at a given moment in time.

(a) Show that x satisfies the equation of motion $\mathrm{d}^2 x/\mathrm{d}t^2 = gx/L$. [Hint: Use $F = \mathrm{d}p/\mathrm{d}t$, which allows you to handle the two parts of the rope separately even though mass is moving out of one part and into the other.]

(b) Give a physical explanation for the fact that a larger value of x on the right-hand side of the equation leads to a greater value of the acceleration on the left side.

(c) When we take the second derivative of the function $x(t)$ we are supposed to get essentially the same function back again, except for a constant out in front. The function e^x has the property that it is unchanged by differentiation, so it is reasonable to look for solutions to this problem that are of the form $x = be^{ct}$, where b and c are constants. Show that this does indeed provide a solution for two specific values of c (and for any value of b).

(d) Show that the sum of any two solutions to the equation of motion is also a solution.

(e) Find the solution for the case where the rope starts at rest at $t = 0$ with some nonzero value of x.

⋆

8-m14 A rocket ejects exhaust with an exhaust velocity u. The rate at which the exhaust mass is used (mass per unit time) is b. We assume that the rocket accelerates in a straight line starting from rest, and that no external forces act on it. Let the rocket's initial mass (fuel plus the body and payload) be m_i, and m_f be its final mass, after all the fuel is used up. (a) Find the rocket's final velocity, v, in terms of u, m_i, and m_f. Neglect the effects of special relativity. (b) A typical exhaust velocity for chemical rocket engines is 4000 m/s. Estimate the initial mass of a rocket that could accelerate a one-ton payload to 10% of the speed of light, and show that this design won't work. (For the sake of the estimate, ignore the mass of the fuel tanks. The speed is

fairly small compared to c, so it's not an unreasonable approximation to ignore relativity.)

$\checkmark$ $\star$

9 Conservation of angular momentum

This is not a textbook. It's a book of problems meant to be used along with a textbook. Although each chapter of this book starts with a brief summary of the relevant physics, that summary is not meant to be enough to allow the reader to actually learn the subject from scratch. The purpose of the summary is to show what material is needed in order to do the problems, and to show what terminology and notation are being used.

9.1 Angular momentum

We have discarded Newton's laws of motion and begun the process of rebuilding the laws of mechanics from scratch using conservation laws. So far we have encountered conservation of energy and momentum.

It is not hard to come up with examples to show that this list of conservation laws is incomplete. The earth has been rotating about its own axis at very nearly the same speed (once every 24 hours) for all of human history. Why hasn't the planet's rotation slowed down and come to a halt? Conservation of energy doesn't protect us against this unpleasant scenario. The kinetic energy tied up in the earth's spin could be transformed into some other type of energy, such as heat. Nor is such a deceleration prevented by conservation of momentum, since the total momentum of the earth due to its rotation cancels out.

There is a third important conservation law in mechanics, which is conservation of *angular momentum*. A spinning body such as the earth has angular momentum. Conservation of angular momentum arises from symmetry of the laws of physics with respect to rotation. That is, there is no special direction built into the laws of physics, such as the direction toward the constellation Sagittarius. Suppose that a non-spinning asteroid were to gradually start spinning. Even if there were some source of energy to initiate this spin (perhaps the heat energy stored in the rock), it wouldn't make sense for the spin to start spontaneously, because then the axis of spin would have to point in some direction, but there is no way to determine why one particular direction would be preferred.

As a concrete example, suppose that a bike wheel of radius r and mass m is spinning at a rate such that a point on the rim (where all the mass is concentrated) moves at speed v. We then define the wheel's angular momentum L to have magnitude mvr.

More generally, we define the angular momentum of a system of particles to be the sum of the quantity $\mathbf{r} \times \mathbf{p}$, where $\mathbf{r}$ is the position of a particle relative to an arbitrarily chosen point called the axis, $\mathbf{p}$ is the particle's momentum vector, and $\times$ represents the vector cross product. This definition is chosen both because experiments show that this is the quantity that is conserved. It follows from the definition that angular momentum is a vector, and that its direction is defined by the same right-hand rule used to define the cross product.

9.2 Rigid-body dynamics

In the special case where a rigid body rotates about an axis of symmetry and its center of mass is at rest, the dimension parallel to the axis becomes irrelevant both kinematically and dynamically. We can imagine squashing the system flat so that the object rotates in a two-dimensional plane about a fixed point. The kinetic energy in this situation is

$$K = \frac{1}{2} I \omega^2$$

and the angular momentum about the center of mass is

$$L = I\omega,$$

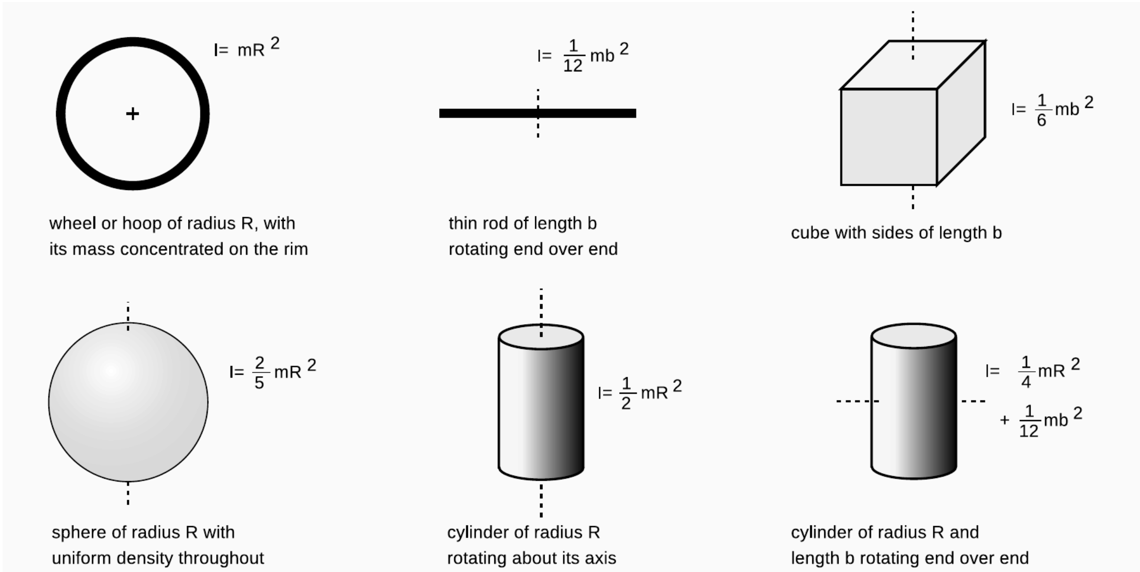


Figure 9.1: Moments of inertia of some geometric shapes.

where I is a constant of proportionality, called the *moment of inertia*. These relations are analogous to $K = (1/2)mv^2$ and $\mathbf{p} = m\mathbf{v}$ for motion of a particle. The angular velocity can also be made into a vector $\boldsymbol{\omega}$, which points along the axis in the right-handed direction. We then have $\mathbf{L} = I\boldsymbol{\omega}$.

For a system of particles, the moment of inertia is given by

$$I = \sum m_i r_i^2,$$

where r_i is the ith particle's distance from the axis. For a continuous distribution of mass,

$$I = \int r^2 \, \mathrm{d}m.$$

Figure **??** gives the moments of inertia of some commonly encountered shapes.

9.3 Torque

For each of the three conserved quantities we have encountered so far, we can define a rate of transfer or transformation:

$$P = \frac{\mathrm{d}E}{\mathrm{d}t} \qquad \text{[power]}$$

$$\mathbf{F} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} \qquad \text{[force]}$$

$$\boldsymbol{\tau} = \frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} \qquad \text{[torque].}$$

The torque exerted by a force $\mathbf{F}$ can be expressed as

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}.$$

For rigid-body rotation about an axis of symmetry, we have

$$\alpha = \frac{\boldsymbol{\tau}}{I},$$

which is analogous to Newton's second law.

9.4 Statics

If an object is to be in *equilibrium*, both the total torque and the total force acting on it must be

zero:

$$\sum \mathbf{F} = 0$$
$$\sum \boldsymbol{\tau} = 0$$

We consider an object to be in equilibrium when these conditions apply, even if it is moving.

An equilibrium can be stable, like a marble at the bottom of a bowl, unstable, like a marble placed on top of a hemispherical hill, or neutral, like a marble placed on a flat tabletop.

Problems

9-a1 A skilled motorcyclist can ride up a ramp, fly through the air, and land on another ramp. Why would it be useful for the rider to speed up or slow down the back wheel while in the air?

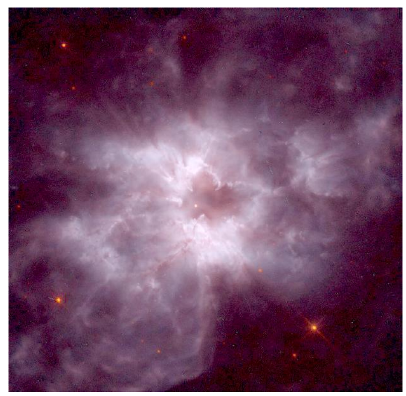
9-a2 The earth moves about the sun in an elliptical orbit where angular momentum about the sun is conserved. Earth's distance from the sun ranges from 0.983 AU (at perihelion, the closest point) to 1.017 AU (at aphelion, the farthest point). The AU is a unit of distance. If the orbital speed of Earth at aphelion is v, what is the orbital speed of Earth at perihelion?

9-a3 (a) Alice says Cathy's body has zero momentum, but Bob says Cathy's momentum is nonzero. Nobody is lying or making a mistake. How is this possible? Give a concrete example.
(b) Alice and Bob agree that Dong's body has nonzero momentum, but disagree about Dong's angular momentum, which Alice says is zero, and Bob says is nonzero. Explain.

9-a4 Two objects have the same momentum vector. Assume that they are not spinning; they only have angular momentum due to their motion through space. Can you conclude that their angular momenta are the same? Explain. [Based on a problem by Serway and Faughn.]

9-a5 Find the angular momentum of a particle whose position is $\mathbf{r} = 3\hat{\mathbf{x}} - \hat{\mathbf{y}} + \hat{\mathbf{z}}$ (in meters) and whose momentum is $\mathbf{p} = -2\hat{\mathbf{x}} + \hat{\mathbf{y}} + \hat{\mathbf{z}}$ (in kg·m/s). ✓

9-d1 The sun turns on its axis once every 26.0 days. Its mass is 2.0×10^{30} kg and its radius is 7.0×10^{8} m. Assume it is a rigid sphere of uniform density.
(a) What is the sun's angular momentum? ✓
In a few billion years, astrophysicists predict that the sun will use up all its sources of nuclear energy, and will collapse into a ball of exotic, dense matter known as a white dwarf. Assume that its radius becomes 5.8×10^{6} m (similar to the size of the Earth.) Assume it does not lose any mass between now and then. (Don't be fooled by the photo, which makes it look like nearly all of the star was thrown off by the explosion. The visually prominent gas cloud is actually thinner than the best laboratory vacuum ever produced on earth. Certainly a little bit of mass is actually lost, but it is not at all unreasonable to make an approximation of zero loss of mass as we are doing.)
(b) What will its angular momentum be?
(c) How long will it take to turn once on its axis? ✓



Problem 9-d1.

9-d2 Give a numerical comparison of the two molecules' moments of inertia for rotation in the plane of the page about their centers of mass. ✓

9-d3 A baseball pitcher can throw a curveball toward home plate at 138 km/hr with a spin of 2500 r.p.m. What percentage of the total KE of the baseball is in rotational kinetic energy? Treat the 145-gram baseball as a uniform sphere of radius 3.7 cm. ✓

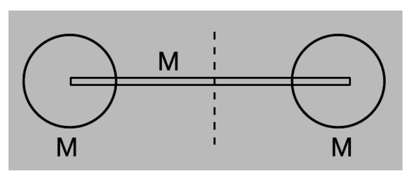
9-d4 A merry-go-round consists of a uniform disc of mass M. It spins around at N revolutions per minute. Mary, who also has mass M, is thinking about jumping on.
(a) What is the initial angular velocity of the merry-go-round (in terms of N alone)? $\checkmark$
(b) Suppose Mary jumps on the merry-go-round at the edge with negligible initial velocity. How many revolutions per minute does the merry-go-round make? $\checkmark$
(c) Suppose that, instead of jumping on the edge with no initial velocity, Mary wants to jump onto the edge (tangent to the disc) such that the merry-go-round is at rest right after jumping on. Express your answer in terms of the radius of the merry-go-round, R, and the initial angular velocity of the merry-go-round, ω.
$\checkmark$

9-d5 A circular, solid disc of radius R and mass $2M$ is rotating with angular velocity ω. A second disc of radius R and mass M is dropped onto the rotating disc, and the two slide against each other until they reach the same final angular velocity.
(a) What is the final angular velocity of the discs? $\checkmark$
(b) What percentage of the initial KE was lost during the collision?
$\checkmark$

9-d6 In a physics lecture, a student holds a bicycle wheel of moment of inertia I while sitting on a stool that can spin. With her feet on the ground so as not to move, she starts the wheel spinning with angular velocity ω in the counterclockwise direction when looking down from above, so that the angular velocity vector points towards the ceiling. She then picks up her feet and turns the wheel over so that the rotation is reversed. This causes her to start rotating about the stool's axis of rotation. In the following, take angular momenta to be positive if pointing vertically upwards and negative if pointing vertically downwards.
(a) What is the final (spin) angular momentum of the bicycle wheel? $\checkmark$
(b) What is the final angular momentum of rest of the system, assuming there is no external frictional torque during the flip? $\checkmark$

9-d7 Show that a sphere of radius R that is rolling without slipping has angular momentum and momentum in the ratio $L/p = (2/5)R$.

9-d8 A dumbbell consists of two solid, spherical masses (each of mass M) attached to the two ends of a bar, also of mass M. The length of the bar is L.
(a) Find the moment of inertia of the mass distribution about an axis through the center of mass and perpendicular to the bar, assuming the balls are pointlike. $\checkmark$
(b) Now find the moment of inertia assuming the spherical masses have radius r. You will need to use the parallel-axis theorem for this. To simplify the math and make it easier to compare the answers to parts a and b, take the densities to add in the places where the bar's volume overlaps with the volume of a ball. $\checkmark$
(c) Let $r = L/5$. By what percentage is your result from part a off from that in part b? Give your answer to two significant figures.
$\checkmark$



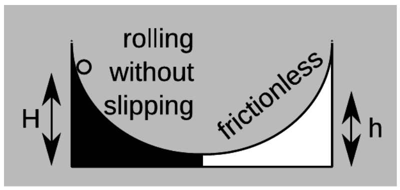
Problem 9-d8.

9-d9 A solid sphere of weight W rolls without slipping up an incline at an angle of θ (with respect to the horizontal). At the bottom of the incline the center of mass of the sphere has translational speed v. How far along the ramp does the sphere travel before coming to rest (not vertical height, but distance along the ramp)?
$\checkmark$

9-d10 A sphere of mass M and radius R is not necessarily solid or hollow. It has moment of

inertia $I = cMR^2$. As shown in the figure, the sphere starts from rest and rolls without slipping down a ramp from height H. It then moves back up the other side, but now with no friction at all between the sphere and the ramp. What height does the sphere reach?

√



Problem 9-d10.

9-d11 You race a hoop, a solid sphere, and a solid cylinder down an incline of angle θ with respect to the horizontal. Each object rolls without slipping.

(a) What is the linear acceleration of the center of mass of the hoop? √

(b) The solid sphere? √

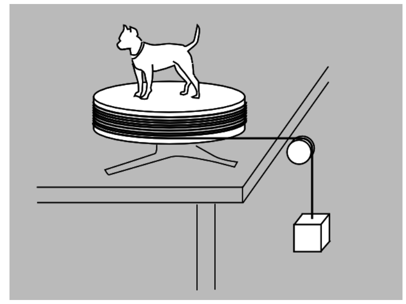
(c) The solid cylinder?Note this problem is easier if you don't do each part separately, but rather say that $I = cMR^2$, and plug in different values of c at the very end of the calculation.

√

9-d12 The figure shows a tabletop experiment that can be used to determine an unknown moment of inertia. A rotating platform of radius R has a string wrapped around it. The string is threaded over a pulley and down to a hanging weight of mass m. The mass is released from rest, and its downward acceleration a ($a > 0$) is measured. Find the total moment of inertia I of the platform plus the object sitting on top of it. (The moment of inertia of the object itself can then be found by subtracting the value for the empty platform.)

√

9-d13 Show that when a thin, uniform ring rotates about a diameter, the moment of inertia is half as big as for rotation about the axis of symmetry.



Problem 9-d12.

▷ Solution, p. 202

9-e1 The nucleus ^{168}Er (erbium-168) contains 68 protons (which is what makes it a nucleus of the element erbium) and 100 neutrons. It has an ellipsoidal shape like an American football, with one long axis and two short axes that are of equal diameter. Because this is a subatomic system, consisting of only 168 particles, its behavior shows some clear quantum-mechanical properties. It can only have certain energy levels, and it makes quantum leaps between these levels. Also, its angular momentum can only have certain values, which are all multiples of 2.109×10^{-34} kg · m^2/s. The table shows some of the observed angular momenta and energies of ^{168}Er, in SI units (kg · m^2/s and joules).

$L \times 10^{34}$	$E \times 10^{14}$
0	0
2.109	1.2786
4.218	4.2311
6.327	8.7919
8.437	14.8731
10.546	22.3798
12.655	31.135
14.764	41.206
16.873	52.223

(a) These data can be described to a good approximation as a rigid end-over-end rotation. Estimate a single best-fit value for the moment of inertia from the data, and check how well the data agree with the assumption of rigid-body rotation. √

(b) Check whether this moment of inertia is on

the right order of magnitude. The moment of inertia depends on both the size and the shape of the nucleus. For the sake of this rough check, ignore the fact that the nucleus is not quite spherical. To estimate its size, use the fact that a neutron or proton has a volume of about 1 fm^3 (one cubic femtometer, where 1 fm $= 10^{-15}$ m), and assume they are closely packed in the nucleus.

9-e2 When we talk about rigid-body rotation, the concept of a perfectly rigid body can only be an idealization. In reality, any object will compress, expand, or deform to some extent when subjected to the strain of rotation. However, if we let it settle down for a while, perhaps it will reach a new equilibrium. As an example, suppose we fill a centrifuge tube with some compressible substance like shaving cream or Wonder Bread. We can model the contents of the tube as a one-dimensional line of mass, extending from $r = 0$ to $r = \ell$. Once the rotation starts, we expect that the contents will be most compressed near the "floor" of the tube at $r = \ell$; this is both because the inward force required for circular motion increases with r for a fixed ω, and because the part at the floor has the greatest amount of material pressing "down" (actually outward) on it. The linear density $\mathrm{d}m/\,\mathrm{d}r$, in units of kg/m, should therefore increase as a function of r. Suppose that we have $\mathrm{d}m/\,\mathrm{d}r = \mu e^{r/\ell}$, where μ is a constant. Find the moment of inertia.

✓

9-e3 (a) As suggested in the figure, find the area of the infinitesimal region expressed in polar coordinates as lying between r and $r+\mathrm{d}r$ and between θ and $\theta+\mathrm{d}\theta$. ✓

(b) Generalize this to find the infinitesimal element of volume in cylindrical coordinates (r,θ,z), where the Cartesian z axis is perpendicular to the directions measured by r and θ. ✓

(c) Find the moment of inertia for rotation about its axis of a cone whose mass is M, whose height is h, and whose base has a radius b.

✓

9-e4 Find the moment of inertia of a solid rectangular box of mass M and uniform density, whose sides are of length a, b, and c, for rotation about an axis through its center parallel to the edges of length a.

✓

9-e5 (a) Prove the identity $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \mathbf{b}(\mathbf{a}\cdot\mathbf{c}) - \mathbf{c}(\mathbf{a}\cdot\mathbf{b})$ by expanding the product in terms of its components. Note that because the x, y, and z components are treated symmetrically in the definitions of the vector cross product, it is only necessary to carry out the proof for the x component of the result.

(b) Applying this to the angular momentum of a rigidly rotating body, $L = \int \mathbf{r}\times(\boldsymbol{\omega}\times\mathbf{r})\,\mathrm{d}m$, show that the diagonal elements of the moment of inertia tensor can be expressed as, e.g., $I_{xx} = \int(y^2+z^2)\,\mathrm{d}m$.

(c) Find the diagonal elements of the moment of inertia matrix of an ellipsoid with axes of lengths a, b, and c, in the principal-axis frame, and with the axis at the center.

✓

9-e6 Let two sides of a triangle be given by the vectors $\mathbf{A}$ and $\mathbf{B}$, with their tails at the origin, and let mass m be uniformly distributed on the interior of the triangle. (a) Show that the distance of the triangle's center of mass from the intersection of sides $\mathbf{A}$ and $\mathbf{B}$ is given by $\frac{1}{3}|\mathbf{A}+\mathbf{B}|$.

(b) Consider the quadrilateral with mass $2m$, and vertices at the origin, $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{A}+\mathbf{B}$. Show that its moment of inertia, for rotation about an axis perpendicular to it and passing through its center of mass, is $\frac{m}{6}(A^2+B^2)$.

(c) Show that the moment of inertia for rotation about an axis perpendicular to the plane of the original triangle, and passing through its center of mass, is $\frac{m}{18}(A^2+B^2-\mathbf{A}\cdot\mathbf{B})$. Hint: Combine the results of parts a and b with the result of problem **??**.

9-e7 In this problem we investigate the notion of division by a vector.

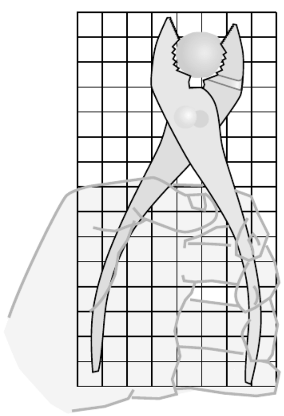
(a) Given a nonzero vector $\mathbf{a}$ and a scalar b, suppose we wish to find a vector $\mathbf{u}$ that is the solution of $\mathbf{a}\cdot\mathbf{u} = b$. Show that the solution is

not unique, and give a geometrical description of the solution set.
(b) Do the same thing for the equation $\mathbf{a}\times\mathbf{u} = \mathbf{c}$.
(c) Show that the *simultaneous* solution of these two equations exists and is unique.

Remark: This is one motivation for constructing the number system called the quaternions. For a certain period around 1900, quaternions were more popular than the system of vectors and scalars more commonly used today. They still have some important advantages over the scalar-vector system for certain applications, such as avoiding a phenomenon known as gimbal lock in controlling the orientation of bodies such as spacecraft.

⋆

9-g1 The figure shows scale drawing of a pair of pliers being used to crack a nut, with an appropriately reduced centimeter grid. Warning: do not attempt this at home; it is bad manners. If the force required to crack the nut is 300 N, estimate the force required of the person's hand.
▷ Solution, p. 202



Problem 9-g1.

9-g2 Make a rough estimate of the mechanical advantage of the lever shown in the figure. In other words, for a given amount of force applied on the handle, how many times greater is the resulting force on the cork?

Problem 9-g2.

9-g3 An object is observed to have constant angular momentum. Can you conclude that no torques are acting on it? Explain. [Based on a problem by Serway and Faughn.]

9-g4 You are trying to loosen a stuck bolt on your RV using a big wrench that is 50 cm long. If you hang from the wrench, and your mass is 55 kg, what is the maximum torque you can exert on the bolt?

√

9-g5 A bicycle wheel with moment of inertia 0.15 kg·m^2 takes 30 seconds to come to rest from an initial angular velocity of 90 r.p.m. What is the magnitude of the average frictional torque over this deceleration?

9-g6 The graph shows the torque on a rigid body as a function of time. The body is at rest at $t = 2$ s. Later, at $t = 5$ s, it is spinning at angular velocity 90 s^{-1}.
(a) What is the moment of inertia of the body?

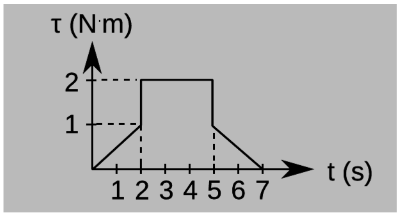
√

(b) What is its angular velocity at $t = 7$ s?

√

9-g7 A disk starts from rest and rotates about a fixed axis, subject to a constant torque. The work done by the torque during the first revolution is W. What is the work done by the torque during the second revolution?

√



Problem 9-g6.

9-g8 A physical therapist wants her patient to rehabilitate his injured elbow by laying his arm flat on a table, and then lifting a 2.1 kg mass by bending his elbow. In this situation, the weight is 33 cm from his elbow. He calls her back, complaining that it hurts him to grasp the weight. He asks if he can strap a bigger weight onto his arm, only 17 cm from his elbow. How much mass should she tell him to use so that he will be exerting the same torque? (He is raising his forearm itself, as well as the weight.) ✓

9-g9 Two horizontal tree branches on the same tree have equal diameters, but one branch is twice as long as the other. Give a quantitative comparison of the torques where the branches join the trunk. [Thanks to Bong Kang.]

9-g10 Penguins are playful animals. Tux the Penguin invents a new game using a natural circular depression in the ice. He waddles at top speed toward the crater, aiming off to the side, and then hops into the air and lands on his belly just inside its lip. He then belly-surfs, moving in a circle around the rim. The ice is frictionless, so his speed is constant. Is Tux's angular momentum zero, or nonzero? What about the total torque acting on him? Take the center of the crater to be the axis. Explain your answers.

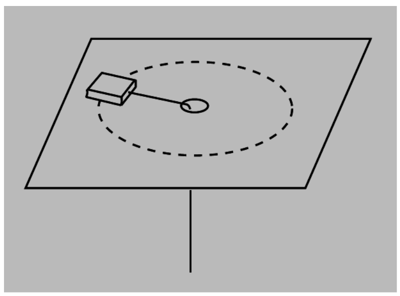
9-j1 A massless rod of length ℓ has weights, each of mass m, attached to its ends. The rod is initially put in a horizontal position, and laid on an off-center fulcrum located at a distance b from the rod's center. The rod will topple. (a) Calculate the total gravitational torque on the rod directly, by adding the two torques. (b) Verify that this gives the same result as would have been obtained by taking the entire gravitational force as acting at the center of mass.

9-j2 A solid rectangular door of moment of inertia I is initially open and at rest when a piece of sticky clay of mass m and velocity v_0 strikes the door perpendicularly at a distance d from the axis of the hinges.
(a) Find the angular speed of the door just after the sticky collision. ✓
(b) What fraction of the initial KE was lost during the sticky collision. ✓
(c) Suppose the door comes to rest after rotating $\Delta\theta > 0$ because of a constant frictional torque. What is the value of this frictional torque? ✓

9-j3 A ball is connected by a string to a vertical post. The ball is set in horizontal motion so that it starts winding the string around the post. Assume that the motion is confined to a horizontal plane, i.e., ignore gravity. Michelle and Astrid are trying to predict the final velocity of the ball when it reaches the post. Michelle says that according to conservation of angular momentum, the ball has to speed up as it approaches the post. Astrid says that according to conservation of energy, the ball has to keep a constant speed. Who is right? [Hint: How is this different from the case where you whirl a rock in a circle on a string and gradually reel in the string?]

9-j4 A book is spinning around a circle of radius R on top of a flat tabletop. The speed of the book is initially v_0. The inward force is provided by a string attached to the book, going through a hole at the center of the circle ($r = 0$), and passing underneath to a person holding the string. The person pulls on the string so that the book spirals inward, eventually cutting the radius of the circular motion in half.
(a) What is the speed of the book at $r = R/2$? ✓
(b) How much work is done by the person pulling

on the string as the book moves from $r = R$ to $r = R/2$? ✓
(The string's force exerts no torque on the book, since it is always in the direction towards the hole. The force, however, can have a non-zero component tangential to the book's motion, because the book spirals in toward the hole. This is why the answer to part b is nonzero.)

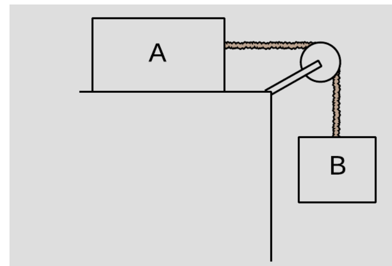


Problem 9-j4.

9-j5 Suppose a bowling ball is initially thrown so that it has no angular momentum at all, i.e., it is initially just sliding down the lane. Eventually kinetic friction will get it spinning fast enough so that it is rolling without slipping. Show that the final velocity of the ball equals 5/7 of its initial velocity. [Hint: You'll need the result of problem 9-d7.]

9-j6 A yo-yo of total mass m consists of two solid cylinders of radius R, connected by a small spindle of negligible mass and radius r. The top of the string is held motionless while the string unrolls from the spindle. Show that the acceleration of the yo-yo is $g/(1 + R^2/2r^2)$. [Hint: The acceleration and the tension in the string are unknown. Use $\tau = \Delta L/\Delta t$ and $F = ma$ to determine these two unknowns.]

9-m1 Block A (of mass M_A) rests on a frictionless horizontal table. It is connected via a light string to block B (of mass M_B) hanging over the edge of the table. The pulley itself, a solid disc, has non-negligible mass M_C. The light string does not slip over the pulley. What is the magnitude of the linear acceleration of the system?



Problem 9-m1.

9-m2 A rod of length b and mass m stands upright. We want to strike the rod at the bottom, causing it to fall and land flat.
(a) Find the momentum, p, that should be delivered, in terms of m, b, and g. [Hint: the solution is nearly determined by units.] ✓
(b) Can this really be done without having the rod scrape on the floor?

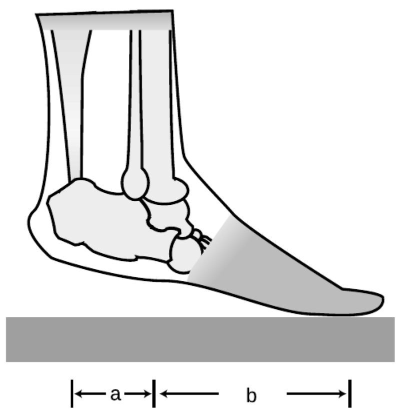
⋆

9-m3 (a) Find the moment of inertia of a uniform square of mass m and with sides of length b, for rotation in its own plane, about one of its corners. ✓
(b) The square is balanced on one corner on a frictionless surface. An infinitesimal perturbation causes it to topple. Find its angular velocity at the moment when its side slaps the surface.
✓ ⋆

9-p1 An object thrown straight up in the air is momentarily at rest when it reaches the top of its motion. Does that mean that it is in equilibrium at that point? Explain.

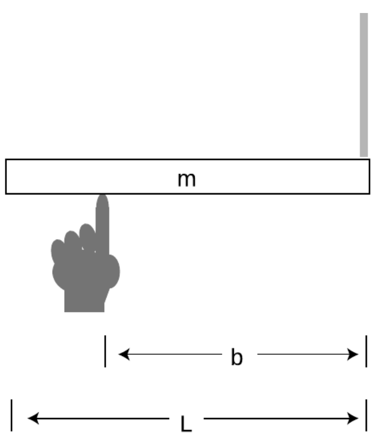
9-p2 A person of weight W stands on the ball of one foot. Find the tension in the calf muscle and the force exerted by the shinbones on the bones of the foot, in terms of W, a, and b. For simplicity, assume that all the forces are at 90-degree angles to the foot, i.e., neglect the angle between the foot and the floor.

√



Problem 9-p2.

9-p3 The rod in the figure is supported by the finger and the string.
(a) Find the tension, T, in the string, and the force, F, from the finger, in terms of m, b, L, and g. √
(b) Comment on the cases $b = L$ and $b = L/2$.
(c) Are any values of b unphysical?



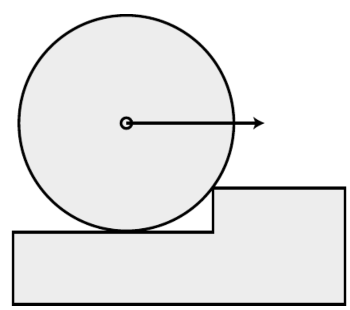
Problem 9-p3.

9-p4 Two atoms will interact through electrical forces between their protons and electrons. One fairly good approximation to the electrical energy is the Lennard-Jones formula,

$$U(r) = k\left[\left(\frac{a}{r}\right)^{12} - 2\left(\frac{a}{r}\right)^{6}\right],$$

where r is the center-to-center distance between the atoms and k is a positive constant. Show that (a) there is an equilibrium point at $r = a$,
(b) the equilibrium is stable, and
(c) the energy required to bring the atoms from their equilibrium separation to infinity is k.

9-s1 (a) Find the minimum horizontal force which, applied at the axle, will pull a wheel over a step. Invent algebra symbols for whatever quantities you find to be relevant, and give your answer in symbolic form. [Hints: There are four forces on the wheel at first, but only three when it lifts off. Normal forces are always perpendicular to the surface of contact. Note that the corner of the step cannot be perfectly sharp, so the surface of contact for this force really coincides with the surface of the wheel.]
(b) Under what circumstances does your result become infinite? Give a physical interpretation.

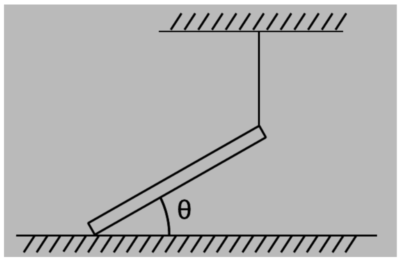


Problem 9-s1.

9-s2 A uniform bar of mass M and length L is held up by a vertical rope attached to the ceiling, as shown.
(a) What is the tension in the rope? √
(b) What is the normal force provided by the

ground on the bar? ✓
(c) What is the static frictional force provided by the ground on the bar?

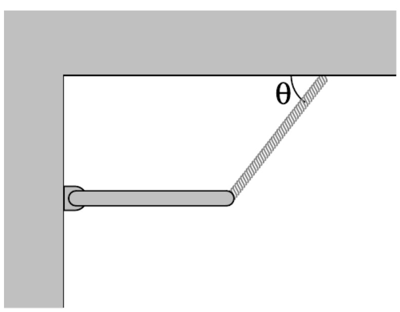
✓



Problem 9-s2.

9-s3 (a) The bar of mass m is attached at the wall with a hinge, and is supported on the right by a massless cable. Find the tension, T, in the cable in terms of the angle θ. ✓
(b) Interpreting your answer to part a, what would be the best angle to use if we wanted to minimize the strain on the cable?
(c) Again interpreting your answer to part a, for what angles does the result misbehave mathematically? Interpet this physically.

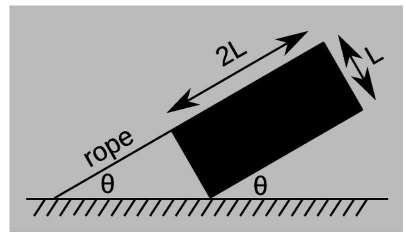
⋆



Problem 9-s3.

9-s4 A block of mass m (uniformly distributed) and dimensions $2L \times L$ is prevented from toppling over by a rope, making the same angle with respect to the horizontal as the long end of the block. ($\theta < \arctan(2) \approx 63.4°$, which is another way of saying that the block's center of mass lies to the right of the point of contact with the ground.) (a) What is the tension in the rope? ✓
(b) What is the normal force provided by the ground? ✓
(c) What is the minimum coefficient of friction required for this configuration to be in static equilibrium? Evaluate your expression for $\theta = 30°$.

✓ ⋆



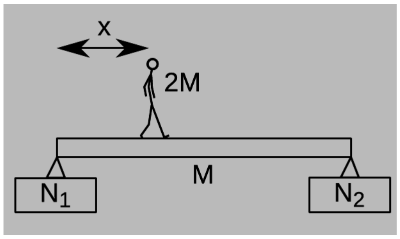
Problem 9-s4.

9-s5 A meter-stick balances on a fulcrum placed at the 50.0 cm mark only when a 4.0 g weight is placed at the 80.0 cm mark. Without the weight, the fulcrum needs to be placed at the 48.0 cm mark to balance the stick. What is the mass of the meter-stick?

9-s6 Julia, whose mass is $2M$, is walking across a uniform bar of mass M and length L, as shown. The bar is supported by scales at its left and right ends. The readings on the scales are N_1 and N_2, respectively.
(a) What is the maximum distance x from the left end of the bar that she can reach so that N_2 does not exceed Mg? ✓
(b) What is the reading N_1 when she reaches this maximum distance?

✓

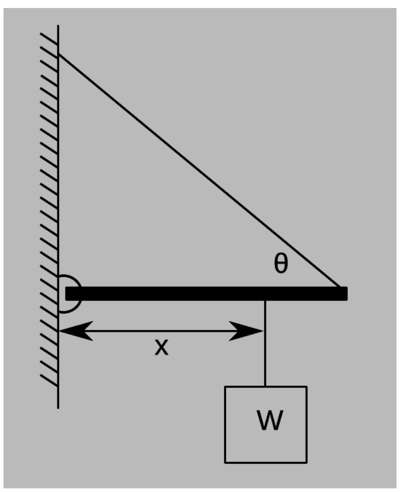
9-s7 A bar of mass m and length L is supported by a hinge on one end and by a rope on the other end. The rope makes an angle θ with respect to the horizontal. In addition, a weight W is hung from the bar at a distance x away from the hinge. Take the $+x$ direction to the right, and $+y$ vertically upwards.



Problem 9-s6.

(a) What is the tension in the rope? ✓
(b) What is the x component of the force from the hinge on the bar? ✓
(c) What is the y component of the force from the hinge on the bar?

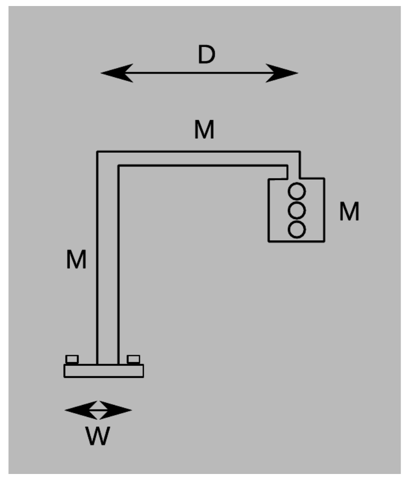
✓



Problem 9-s7.

9-s8 A stoplight consists of three pieces of mass M: a thin, vertical bar mounted at the center of a base of width w, a horizontal bar of length D, and the stoplight fixture itself. (The two bars have a uniform mass distribution.) The structure is screwed to the ground at the base. For simplicity, we take there to be only two screws, one on the left and one on the right side of the base, which has width $w < (D/2)$. Assume that the entire normal force from the ground acts on the right-hand side of the base. (This is where the structure would naturally pivot.)
(a) By taking the right side of the base as your pivot point, you should be able to easily see that the screw on the left must provide a downwards force to keep the stoplight in static equilibrium. What is the magnitude of this force? ✓
(b) Find the upward normal force acting on the base.

✓



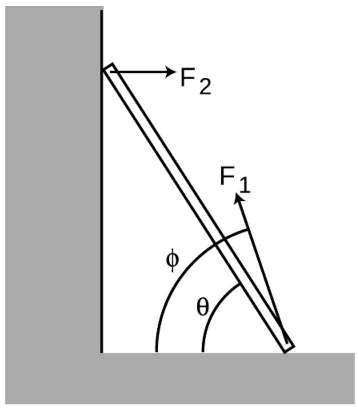
Problem 9-s8.

9-s9 A uniform ladder of mass m and length L leans against a smooth wall, making an angle θ with respect to the ground. The dirt exerts a normal force and a frictional force on the ladder, producing a force vector with magnitude F_1 at an angle ϕ with respect to the ground. Since the wall is smooth, it exerts only a normal force on the ladder; let its magnitude be F_2.
(a) Explain why ϕ must be greater than θ. No math is needed.
(b) Choose any numerical values you like for m and L, and show that the ladder can be in equilibrium (zero torque and zero total force vector) for $\theta = 45.00°$ and $\phi = 63.43°$.

9-v1 Continuing problem 9-s9, find an equation for ϕ in terms of θ, and show that m and

L do not enter into the equation. Do not assume any numerical values for any of the variables. You will need the trig identity $\sin(a-b) = \sin a \cos b - \sin b \cos a$. (As a numerical check on your result, you may wish to check that the angles given in part b of the previous problem satisfy your equation.)

✓ ⋆

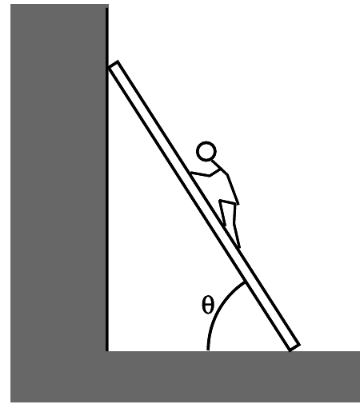


Problem 9-v1.

9-v2 A person of mass M is climbing a ladder of length L and negligible mass, propped up against a wall making an angle θ with respect to the horizontal. There is no friction between the ladder and the wall, but there is a coefficient of static friction μ_s between the ladder and the ground. What is the maximum distance along the ladder that the person can reach before the ladder starts to slide?

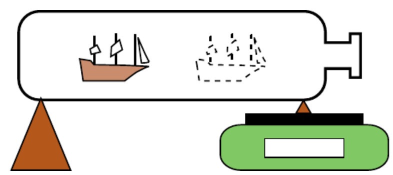
⋆

9-v3 You wish to determine the mass of a ship in a bottle without taking it out. Show that this can be done with the setup shown in the figure, with a scale supporting the bottle at one end, provided that it is possible to take readings with the ship slid to several different locations. Note that you can't determine the position of the ship's center of mass just by looking at it, and likewise for the bottle. In particular, you can't just say, "position the ship right on top of the fulcrum" or "position it right on top of the balance."



Problem 9-v2.

⋆


Problem 9-v3.

9-v4 The box shown in the figure is being accelerated by pulling on it with the rope.

(a) Assume the floor is frictionless. What is the maximum force that can be applied without causing the box to tip over? ✓

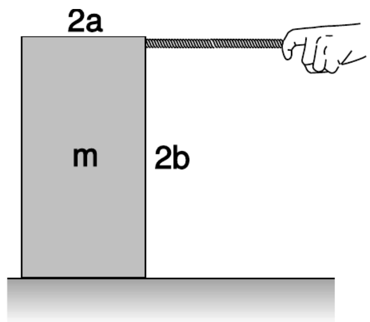
(b) Repeat part a, but now let the coefficient of friction be μ. ✓

(c) What happens to your answer to part b when the box is sufficiently tall? How do you interpret this?

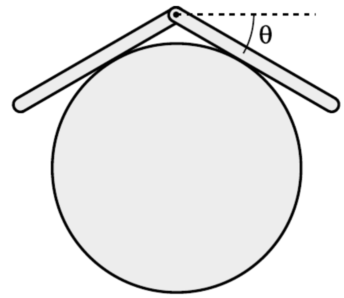
⋆

9-v5 (a) The two identical rods are attached to one another with a hinge, and are supported by the two massless cables. Find the angle α in terms of the angle β, and show that the result is a purely geometric one, independent of the other variables involved. ✓

(b) Using your answer to part a, sketch the configurations for $\beta \to 0$, $\beta = 45°$, and $\beta = 90°$. Do your results make sense intuitively?

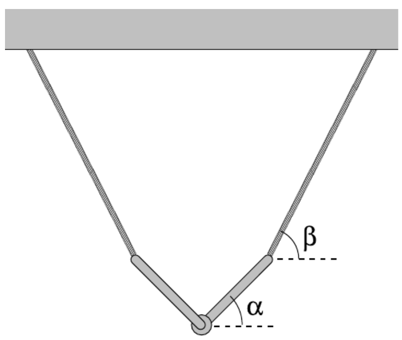


Problem 9-v4.



Problem 9-v6.

⋆



Problem 9-v5.

9-v6 Two bars of length L are connected with a hinge and placed on a frictionless cylinder of radius r. (a) Show that the angle θ shown in the figure is related to the unitless ratio r/L by the equation

$$\frac{r}{L} = \frac{\cos^2\theta}{2\tan\theta}.$$

(b) Discuss the physical behavior of this equation for very large and very small values of r/L.

⋆

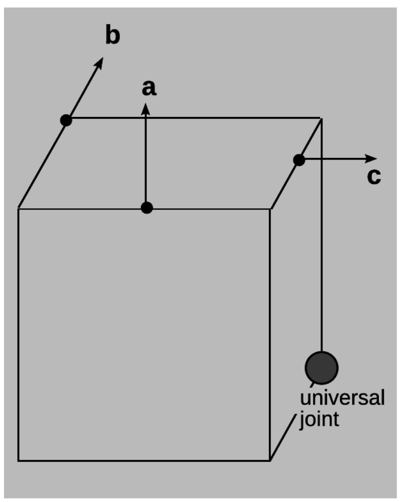
9-v7 We have n identical books of width w, and we wish to stack them at the edge of a table so that they extend the maximum possible distance L_n beyond the edge. Surprisingly, it is possible to have values of L_n that are greater than w, even with fairly small n. For large n, however, L_n begins to grow very slowly. Our goal is to find L_n for a given n. We adopt the restriction that only one book is ever used at a given height.[1] (a) Use proof by induction to find L_n, expressing your result as a sum. (b) Find a sufficiently tight lower bound on this sum, as a closed-form expression, to prove that 1,202,604 books suffice for $L > 7w$.

⋆

9-v8 The uniform cube has unit weight and sides of unit length. One corner is attached to a universal joint, i.e., a frictionless bearing that allows any type of rotation. If the cube is in equilibrium, find the magnitudes of the forces **a**, **b**, and **c**.

√

[1]When this restriction is lifted, the calculation of L_n becomes a much more difficult problem, which was partially solved in 2009 by Paterson, Peres, Thorup, Winkler, and Zwick.



Problem 9-v8.

10 Fluids

This is not a textbook. It's a book of problems meant to be used along with a textbook. Although each chapter of this book starts with a brief summary of the relevant physics, that summary is not meant to be enough to allow the reader to actually learn the subject from scratch. The purpose of the summary is to show what material is needed in order to do the problems, and to show what terminology and notation are being used.

10.1 Statics

Fluids

In physics, the term *fluid* is used to mean either a gas or a liquid. The important feature of a fluid can be demonstrated by comparing with a cube of jello on a plate. The jello is a solid. If you shake the plate from side to side, the jello will respond by shearing, i.e., by slanting its sides, but it will tend to spring back into its original shape. A solid can sustain shear forces, but a fluid cannot. A fluid does not resist a change in shape unless it involves a change in volume.

Pressure

We begin by restricting ourselves to the case of fluid statics, in which the fluid is at rest and in equilibrium. A small chunk or "parcel" of the fluid has forces acting on it from the adjacent portions of the fluid. We have assumed that the parcel is in equilibrium, and if no external forces are present then these forces must cancel. By the definition of a fluid these forces are perpendicular to any part of the imaginary boundary surrounding the parcel. Since force is an additive quantity, the force the fluid exerts on any surface must be proportional to the surface's area. We therefore define the *pressure* to be the (perpendicular) force per unit area,

$$P = \frac{F_\perp}{A}. \tag{10.1}$$

If the forces are to cancel, then this force must be the same on all sides on an object such as a cube, so it follows that pressure has no direction: it is a scalar. The SI units of pressure are newtons per square meter, which can be abbreviated as pascals, 1 Pa = 1 N/m^2. The pressure of the earth's atmosphere at sea level is about 100 kPa.

Only pressure *differences* are ordinarily of any importance. For example, your ears hurt when you fly in an airplane because there is a pressure difference between your inner ear and the cabin; once the pressures are equalized, the pain stops.

Variation of pressure with depth

If a fluid is subjected to a gravitational field and is in equilibrium, then the pressure can only depend on depth (figure 10.1).

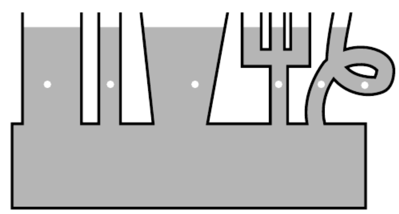


Figure 10.1: The pressure is the same at all the points marked with dots.

To find the variation with depth, we consider the vertical forces acting on a tiny, cubical parcel of the fluid having infinitesimal height dy, where positive y is up. By requiring equilibrium, we find that the difference in pressure between the top and bottom is d$P = -\rho g\,\mathrm{d}y$. A more elegant way of writing this is in terms of a dot product,

$$\mathrm{d}P = \rho\mathbf{g}\cdot\mathrm{d}\mathbf{y} \tag{10.2}$$

which automatically takes care of the plus or minus sign, and avoids any requirements about the coordinate system. By integrating this equation, we can find the change in pressure ΔP corresponding to any change in depth Δy.

Archimedes' principle

A helium balloon or a submarine experiences unequal pressure above and below, due to the variation of pressure with depth. The total force of the surrounding fluid does not vanish, and is called the buoyant force. In a fluid in equilibrium that does not contain any foreign object, any parcel of fluid evidently has its weight canceled out by the buoyant force on it. This buoyant force is unchanged if another object is substituted for the parcel of fluid, so the buoyant force on a submerged object is upward and equal to the weight of the displaced fluid. This is called *Archimedes' principle.*

10.2 Dynamics

Continuity

We now turn to fluid dynamics, eliminating the restriction to cases in which the fluid is at rest and in equilibrium. Mass is conserved, and this constrains the ways in which a fluid can flow. For example, it is not possible to have a piece of pipe with water flowing *out* of it at each end indefinitely. The principle of continuity states that when a fluid flows steadily (so that the velocity at any given point is constant over time), mass enters and leaves a region of space at equal rates.

Liquids are highly incompressible, so that it is often a good approximation to assume that the density is the same everywhere. In the case of incompressible flow, we can frequently relate the rate of steady flow to the cross-sectional area, as in figure 10.2. Because the water is incompressible, the rate at which mass flows through a perpendicular cross-section depends only on the product of the velocity and the cross-sectional area. Therefore as the water falls and accelerates, the cross-sectional area goes down.

Figure 10.2: Due to conservation of mass, the stream of water narrows.

Bernoulli's equation

Consider a parcel of fluid as it flows from one place to another. If it accelerates or decelerates, then its kinetic energy changes. If it rises or falls, its potential energy changes as well. If there is a net change in $KE + PE$, then this must be accomplished through forces from the surrounding fluid. For example, if water is to move uphill at constant speed, then there must be a pressure difference, such as one produced by a pump. Based on these considerations, one can show that along a streamline of the flow,

$$\rho g y + \frac{1}{2}\rho v^2 + P = \text{constant}, \qquad (10.3)$$

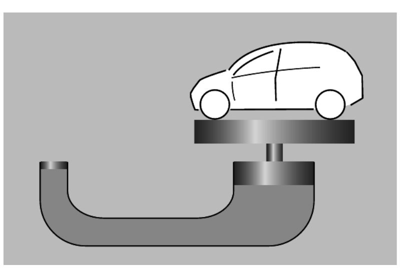
which is *Bernoulli's principle.*

Problems

10-a1 The pressure at 11,000 meters (where most commercial airliners fly) is about a quarter of atmospheric pressure at sea level. Suppose you're a passenger on a plane, flying at such an altitude but with the cabin pressurized to 1 atm, and you notice a gremlin on the wing. Suppose that the window near your seat isn't screwed in and is held in place only by the difference in pressure. What force would the gremlin need to exert on your 40 cm × 50 cm window in order to get in?

√

10-a2 The figure shows a schematic diagram of a car on a hydraulic lift. The small piston has diameter d and the large one D. Any difference in height between the two sides is not enough to create a significant difference in pressure due to gravity.
(a) If we would like to lift a car of mass M, what force is required at the piston end? √
(b) If we would like to lift the car by a distance L, how far does the piston need to move? Assume that the fluid is incompressible, so that its volume is conserved. √
(c) By considering the work done in lifting the car, verify that your answers to part a and b are consistent with conservation of energy.



Problem 10-a2.

10-d1 A drinking straw works because, when you suck air out of the straw, there is decreased pressure. The equilibrium water level inside the straw is such that the pressure at the water level outside the straw (atmospheric pressure) is equal to the pressure at the same height inside the straw. Inside the straw, there are two contributions: the air pressure inside the straw plus the column of water above this height.
(a) Suppose you find that you can only suck water up to a height of 1.0 m. What is the minimum pressure inside the straw? √
(b) Superman, strong as he is, can suck out all the air from a (strong-walled) straw. What is the tallest straw through which Superman can drink water out of a lake?

√

10-d2 The first transatlantic telegraph cable was built in 1858, lying at a depth of up to 3.2 km. What is the pressure at this depth, in atmospheres?

10-d3 One way to measure the density of an unknown liquid is by using it as a barometer. Suppose you have a column of length L of the unknown liquid (inside a vacuum tube), which provides the same pressure as atmospheric pressure P_0.
(a) What is the density of the unknown liquid?

√

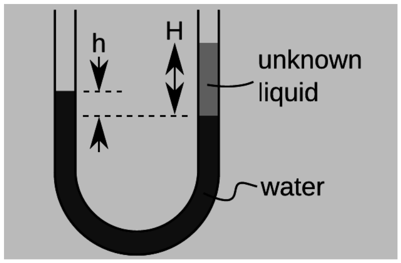
(b) Mercury barometers have $L = 760$ mm at standard atmospheric pressure $P_0 = 1.013 \times 10^5$ Pa. Given these data, what is the density of mercury to three significant figures?

√

10-d4 A U-shaped tube, with both ends exposed to the atmosphere, has two immiscible liquids in it: water, and some unknown liquid. The unknown liquid sits on top of the water on the right side of the tube in a column of height H. Also, the water extends a height h above the unknown-water interface. If the density of water is ρ_w, what is the density of the unknown liquid?

√

10-d5 Typically the atmosphere gets colder with increasing altitude. However, sometimes there is an *inversion layer*, in which this trend is reversed, e.g., because a less dense mass of warm air moves into a certain area, and rises above the



Problem 10-d4.

denser colder air that was already present. Suppose that this causes the pressure P as a function of height y to be given by a function of the form $P = P_o e^{-ky}(1+by)$, where constant temperature would give $b = 0$ and an inversion layer would give $b > 0$. (a) Infer the units of the constants P_o, k, and b. (b) Find the density of the air as a function of y, of the constants, and of the acceleration of gravity g. (c) Check that the units of your answer to part b make sense.

▷ Solution, p. 203

10-d6 Estimate the pressure at the center of the Earth, assuming it is of constant density throughout. The gravitational field g is not constant with repsect to depth. It equals Gmr/b^3 for r, the distance from the center, less than b, the earth's radius. Here m is the mass of the earth, and G is Newton's universal gravitational constant, which has units of $\text{N}\cdot\text{m}^2/\text{kg}^2$.
(a) State your result in terms of G, m, and b. ✓
(b) Show that your answer from part a has the right units for pressure.
(c) Evaluate the result numerically. ✓
(d) Given that the earth's atmosphere is on the order of one thousandth the earth's radius, and that the density of the earth is several thousand times greater than the density of the lower atmosphere, check that your result is of a reasonable order of magnitude.

10-g1 A uniform, solid sphere floats in a liquid of known density. We measure its draft, i.e., the depth to which it is submerged. From this measurement, we want to find the density of the sphere.
(a) Based on units, infer as much as possible about the form of the result.
(b) Find the density of the sphere in terms of the relevant variables. ✓

10-g2 Suppose we want to send a space probe to Venus and have it release a balloon that can float over the landscape and collect data. The venerian atmosphere is hot and corrosive, so it would destroy the kind of mylar or rubber balloon we use on earth. The purpose of this problem is to get a feel for things by estimating whether an aluminum beer can full of helium would float on Venus. Here are some data:

density of atmosphere	67 kg/m^3
density of helium	6.0 kg/m^3
mass of beer can	15 g
volume of beer can	330 cm^3

Find the ratio F_B/F_g of the buoyant force to the can's weight. Does it float? ✓

10-g3 Aluminum and lead have densities $2.8\rho_\text{w}$ and $11.3\rho_\text{w}$, respectively, where ρ_w is the density of water. If the maximum mass of the lead that you can lift while underwater is M, what is the maximum for aluminum? ✓

10-g4 A block of wood is floating in water. It has density $k\rho_\text{w}$, where ρ_w is the density of water. When you apply a downward force of magnitude F, the block becomes fully submerged. What is the mass of the block? ✓

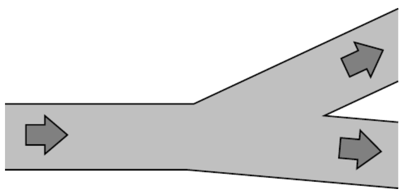
10-g5 Gina has a mass of 62 kg.
(a) Estimate her volume, assuming that her density is the same as that of water. ✓
(b) Gina is in air, which has density 1.2 kg/m^3. What is the buoyant force on her? ✓
(c) Without the air pushing up slightly, she would weigh more on a standard scale. How much more would the scale claim her mass was? ✓

10-g6 An object of mass m has an apparent weight $(3/5)mg$ when submerged in water, which has density ρ_w. If the object is placed

in bromine, which has density $3.1\rho_\mathrm{w}$, it floats. What fraction of the object is submerged in bromine?

$\checkmark$

10-k1 A river with a certain width and depth splits into two parts, each of which has the same width and depth as the original river. What can you say about the speed of the current after the split?



Problem 10-k1.

10-k2 At one cross-section of the Mississippi River, the width is 150 m, the depth is 12 m, and the speed of the water flowing is 1.2 m/s.
(a) What is the rate of flow of the Mississippi River at this location, measured as volume per unit time? $\checkmark$
(b) What is the corresponding rate of flow of mass? $\checkmark$
(c) If the river narrows to 140 m with a depth of 20 m, what is the new speed of the water?

$\checkmark$

10-k3 The firehose shown in the figure has radius a, and water flows through it at speed u. The conical nozzle narrows the radius down to b. Find the speed v at which the water leaves the nozzle.

$\checkmark$

Problem 10-k3.

10-k4 A pipe of slow-moving water is basically at atmospheric pressure, P_0. As the pipe narrows (without changing height), the water speeds up, and the pressure decreases. How fast would the water need to move in order for the pressure in the pipe to be $0.75P_0$?

$\checkmark$

11 Gravity

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11.1 Kepler's laws

Johannes Kepler (1571-1630) studied newly available high-precision data on the motion of the planets, and discovered the following three empirical laws:

Kepler's elliptical orbit law: The planets orbit the sun in elliptical orbits with the sun at one focus.

Kepler's equal-area law: The line connecting a planet to the sun sweeps out equal areas in equal amounts of time.

Kepler's law of periods: The time required for a planet to orbit the sun, called its period, is proportional to the long axis of the ellipse raised to the 3/2 power. The constant of proportionality is the same for all the planets.

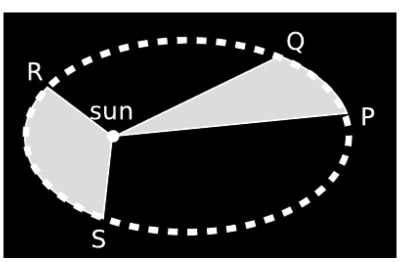


Figure 11.1: If the time interval taken by the planet to move from P to Q is equal to the time interval from R to S, then according to Kepler's equal-area law, the two shaded areas are equal.

11.2 Newton's law of gravity

Kepler's laws were a beautifully simple explanation of what the planets did, but they didn't address why they moved as they did. Once Newton had formulated his laws of motion and taught them to some of his friends, they began trying to connect them to Kepler's laws. It was clear now that an inward force would be needed to bend the planets' paths. Since the outer planets were moving slowly along more gently curving paths than the inner planets, their accelerations were apparently less. This could be explained if the sun's force was determined by distance, becoming weaker for the farther planets. In the approximation of a circular orbit, it is not difficult to show that Kepler's law of periods requires that this weakening with distance vary according to $F \propto 1/r^2$. We know that objects near the earth's surface feel a gravitational force that is also in proportion to their masses. Newton therefore hypothesized a universal law of gravity,

$$F = \frac{Gm_1m_2}{r^2},$$

which states that any two massive particles, anywhere in the universe, attract each other with a certain amount of force. The universal constant G is equal to $6.67 \times 10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2$.

11.3 The shell theorem

Newton proved the following theorem, known as the *shell theorem*:

If an object lies outside a thin, spherical shell of mass, then the vector sum of all the gravitational forces exerted by all the parts of the shell is the same as if the shell's mass had been concentrated at its center. If the object lies inside the shell, then all the gravitational forces cancel out exactly.

The earth is nearly spherical, and the density

in each concentric spherical shell is nearly constant. Therefore for terrestrial gravity, each shell acts as though its mass was at the center, and the result is the same as if the whole mass was there.

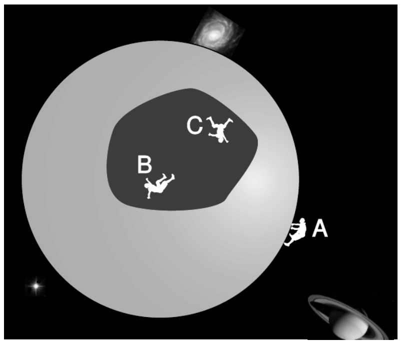


Figure 11.2: Cut-away view of a spherical shell of mass. A, who is outside the shell, feels gravitational forces from every part of the shell — stronger forces from the closer parts, and weaker ones from the parts farther away. The shell theorem states that the vector sum of all the forces is the same as if all the mass had been concentrated at the center of the shell. B, at the center, is clearly weightless, because the shell's gravitational forces cancel out. Surprisingly, C also feels exactly zero gravitational force.

11.4 Universality of free fall

Suppose that masses m_1 and m_2 interact gravitationally, and m_1 is fixed in place, or has so much mass that its inertia makes its acceleration negligible. For example, m_1 could be the earth, and m_2 a rock. When we combine Newton's law of gravity with Newton's second law, we find that m_2's acceleration equals Gm_1/r^2, which is completely independent of the mass m_2. (We assume that no other forces act.) That is, if we give an object a certain initial position and velocity in an ambient gravitational field, then its later motion is independent of its mass: free fall is *universal.*

This fact had first been demonstrated empirically a generation earlier by Galileo, who dropped a cannonball and a musketball simultaneously from the leaning tower of Pisa, and observed that they hit the ground at nearly the same time. This contradicted Aristotle's long-accepted idea that heavier objects fell faster. Modern experiments have verified the universality of free fall to the phenomenal precision of about one part in 10^{11}.

11.5 Current status of Newton's theory

Newton's theory of gravity, according to which masses act on one another *instantaneously* at a distance, is not consistent with Einstein's theory of relativity, which requires that all physical influences travel no faster than the speed of light. Einstein generalized Newton's description of gravity in his general theory of relativity. Newton's theory is a good approximation to the general theory when the masses that interact move at speeds small compared to the speed of light, when the gravitational fields are weak, and when the distances involved are small in cosmological terms. General relativity is needed in order to discuss phenomena such as neutron stars and black holes, the big bang and the expansion of the universe. The effects of general relativity also become important, for example, in the Global Positioning System (GPS), where extremely high precision is required, so that even extremely small deivations from the Newtonian picture are important.

General relativity shares with Newtonian gravity the prediction that free fall is universal. High-precision tests of this universality are therefore stringent tests of both theories.

11.6 Energy

Newton's law of gravity can be restated as a description of energy rather than force. Taking the

indefinite integral of the force law gives the expression

$$PE = -\frac{GMm}{r},$$

where for convenience the constant of integration is taken to be zero.

Problems

11-a1 Astronomers have detected a solar system consisting of three planets orbiting the star Upsilon Andromedae. The planets have been named b, c, and d. Planet b's average distance from the star is 0.059 A.U., and planet c's average distance is 0.83 A.U., where an astronomical unit or A.U. is defined as the distance from the Earth to the sun. For technical reasons, it is possible to determine the ratios of the planets' masses, but their masses cannot presently be determined in absolute units. Planet c's mass is 3.0 times that of planet b. Compare the star's average gravitational force on planet c with its average force on planet b. [Based on a problem by Arnold Arons.]

▷ Solution, p. 203

11-a2 The star Lalande 21185 was found in 1996 to have two planets in roughly circular orbits, with periods of 6 and 30 years. What is the ratio of the two planets' orbital radii?

$\checkmark$

11-d1 Ceres, the largest asteroid in our solar system, is a spherical body with a mass 6000 times less than the earth's, and a radius which is 13 times smaller. If an astronaut who weighs 400 N on earth is visiting the surface of Ceres, what is her weight?

▷ Solution, p. 203

11-d2 (a) A certain vile alien gangster lives on the surface of an asteroid, where his weight is 0.20 N. He decides he needs to lose weight without reducing his consumption of princesses, so he's going to move to a different asteroid where his weight will be 0.10 N. The real estate agent's database has asteroids listed by mass, however, not by surface gravity. Assuming that all asteroids are spherical and have the same density, how should the mass of his new asteroid compare with that of his old one?
(b) Jupiter's mass is 318 times the Earth's, and its gravity is about twice Earth's. Is this consistent with the results of part a? If not, how do you explain the discrepancy?

▷ Solution, p. 203

11-d3 Roy has a mass of 60 kg. Laurie has a mass of 65 kg. They are 1.5 m apart.
(a) What is the magnitude of the gravitational force of the earth on Roy?
(b) What is the magnitude of Roy's gravitational force on the earth?
(c) What is the magnitude of the gravitational force between Roy and Laurie?
(d) What is the magnitude of the gravitational force between Laurie and the sun?

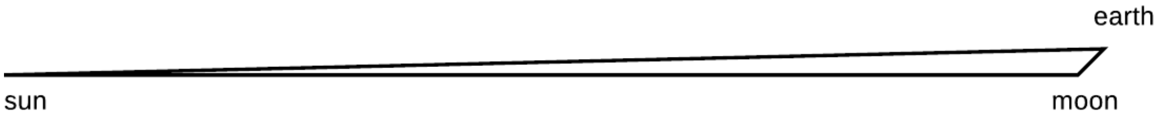
$\checkmark$

11-d4 The planet Uranus has a mass of 8.68×10^{25} kg and a radius of 2.56×10^4 km. The figure shows the relative sizes of Uranus and Earth.
(a) Compute the ratio g_U/g_E, where g_U is the strength of the gravitational field at the surface of Uranus and g_E is the corresponding quantity at the surface of the Earth. $\checkmark$
(b) What is surprising about this result? How do you explain it?

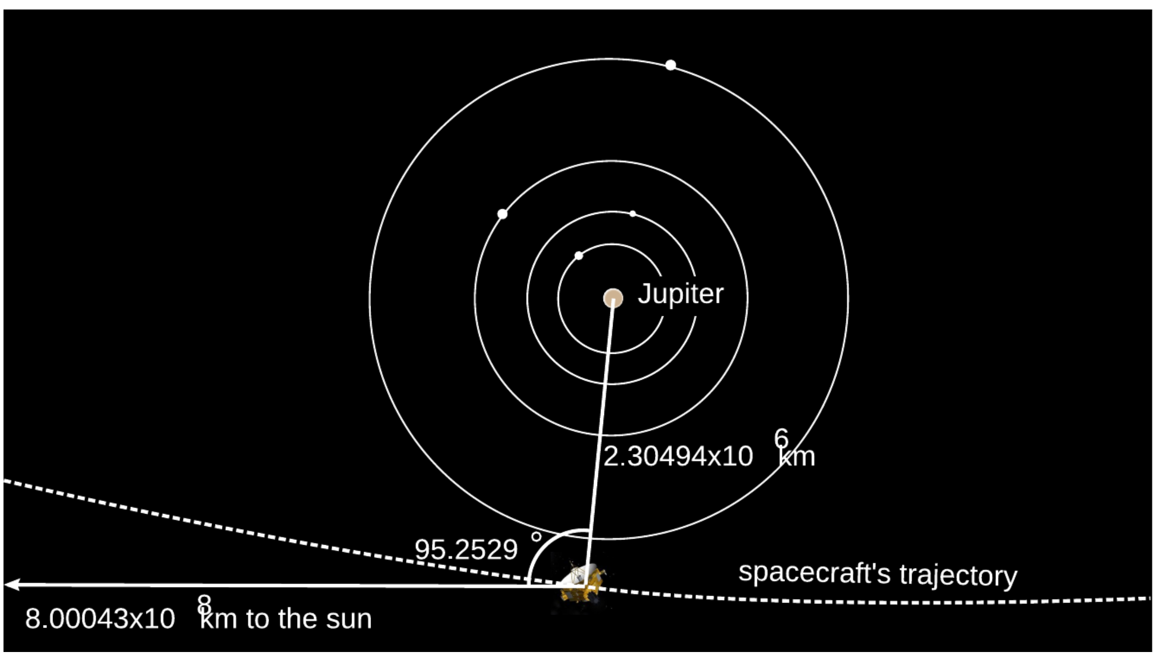
11-d5 The International Space Station orbits at an average altitude of about 370 km above sea level. Compute the value of g at that altitude.

$\checkmark$

11-d6 Two spherical objects, both of mass m and radius R, have center-to-center separation $4R$ and are initially at rest. No external forces act. The spheres accelerate towards one another until they collide.
(a) What is the speed of each object just before the collision? Find the exact answer using energy methods. $\checkmark$
(b) Using the final speed from part (a), and assuming the acceleration is roughly constant and equal to the initial acceleration, calculate an upper limit on the time it takes for the two objects to collide. (The actual time will be shorter, because the acceleration increases as they get closer.) $\checkmark$
(c) Calculate a numerical value of the time from part b, in the case of two bowling balls having $m = 7$ kg and $R = 0.1$ m. $\checkmark$



Problem 11-d9.



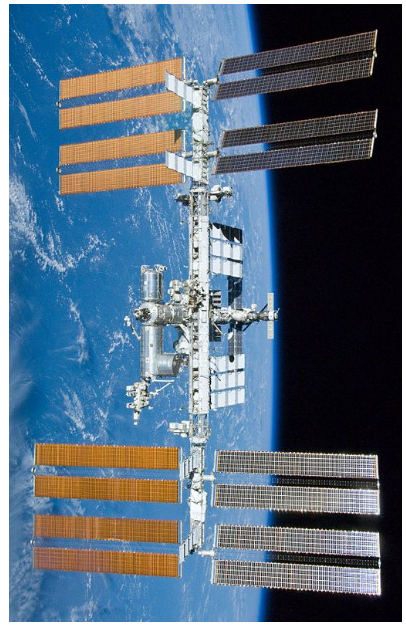
Problem 11-d10.

Problem 11-d4.

11-d7 The figure shows the International Space Station (ISS). One of the purposes of the ISS is supposed to be to carry out experiments in microgravity. However, the following factor limits this application. The ISS orbits the earth once every 92.6 minutes. It is desirable to keep the same side of the station always oriented toward the earth, which means that the station has to rotate with the same period. In the photo, the direction of orbital motion is left or right on the page, so the rotation is about the axis shown as up and down on the page. The greatest distance of any pressurized compartment from the axis of rotation is 36.5 meters. Find the acceleration due to the rotation at this point, and the apparent weight of a 60 kg astronaut at that location. $\checkmark$

11-d8 During a solar eclipse, the moon, earth and sun all lie on the same line, with the moon between the earth and sun. Define your coordinates so that the earth and moon lie at greater x values than the sun. For each force, give the correct sign as well as the magnitude. (a) What force is exerted on the moon by the sun? (b) On the moon by the earth? (c) On the earth by the sun? (d) What total force is exerted on the sun? (e) On the moon? (f) On the earth? $\checkmark$

11-d9 Suppose that on a certain day there is a crescent moon, and you can tell by the shape of the crescent that the earth, sun and moon



Problem 11-d7.

form a triangle with a 135° interior angle at the moon's corner. What is the magnitude of the total gravitational force of the earth and the sun on the moon? (If you haven't done problem 11-d8 already, you might want to try it first, since it's easier, and some of its results can be recycled in this problem.) $\checkmark$

11-d10 On Feb. 28, 2007, the New Horizons space probe, on its way to a 2015 flyby of Pluto, passed by the planet Jupiter for a gravity-assisted maneuver that increased its speed and changed its course. The dashed line in the figure shows the spacecraft's trajectory, which is curved because of three forces: the force of the exhaust gases from the probe's own engines, the sun's gravitational force, and Jupiter's gravitational force. Find the magnitude of the total gravitational force acting on the probe. You will find that the sun's force is much smaller than Jupiter's, so that the magnitude of the total force is determined almost entirely by Jupiter's force. However, this is a high-precision problem, and

you will find that the total force is slightly different from Jupiter's force.

√

11-g1 Tidal interactions with the earth are causing the moon's orbit to grow gradually larger. Laser beams bounced off of a mirror left on the moon by astronauts have allowed a measurement of the moon's rate of recession, which is about 4 cm per year. This means that the gravitational force acting between earth and moon is decreasing. By what fraction does the force decrease with each 27-day orbit?

[Based on a problem by Arnold Arons.]

▷ Solution, p. 203

11-g2 How high above the Earth's surface must a rocket be in order to have 1/100 the weight it would have at the surface? Express your answer in units of the radius of the Earth.

√

11-g3 You are considering going on a space voyage to Mars, in which your route would be half an ellipse, tangent to the Earth's orbit at one end and tangent to Mars' orbit at the other. Your spacecraft's engines will only be used at the beginning and end, not during the voyage. How long would the outward leg of your trip last? (The orbits of Earth and Mars are nearly circular, and Mars's is bigger by a factor of 1.52.)

√

11-g4 Where would an object have to be located so that it would experience zero total gravitational force from the earth and moon?

√

11-j1 In a Star Trek episode, the Enterprise is in a circular orbit around a planet when something happens to the engines. Spock then tells Kirk that the ship will spiral into the planet's surface unless they can fix the engines. Is this scientifically correct? Why?

11-j2 Astronomers have recently observed stars orbiting at very high speeds around an unknown object near the center of our galaxy. For stars orbiting at distances of about 10^{14} m from the object, the orbital velocities are about 10^6 m/s. Assuming the orbits are circular, estimate the mass of the object, in units of the mass of the sun, 2×10^{30} kg. If the object was a tightly packed cluster of normal stars, it should be a very bright source of light. Since no visible light is detected coming from it, it is instead believed to be a supermassive black hole.

√

11-j3 (a) A geosynchronous orbit is one in which the satellite orbits above the equator, and has an orbital period of 24 hours, so that it is always above the same point on the spinning earth. Calculate the altitude of such a satellite. √
(b) What is the gravitational field experienced by the satellite? Give your answer as a percentage in relation to the gravitational field at the earth's surface. √

11-j4 (a) Suppose a rotating spherical body such as a planet has a radius r and a uniform density ρ, and the time required for one rotation is T. At the surface of the planet, the apparent acceleration of a falling object is reduced by the acceleration of the ground out from under it. Derive an equation for the apparent acceleration of gravity, g, at the equator in terms of r, ρ, T, and G. √
(b) Applying your equation from a, by what fraction is your apparent weight reduced at the equator compared to the poles, due to the Earth's rotation? √
(c) Using your equation from a, derive an equation giving the value of T for which the apparent acceleration of gravity becomes zero, i.e., objects can spontaneously drift off the surface of the planet. Show that T only depends on ρ, and not on r. √
(d) Applying your equation from c, how long would a day have to be in order to reduce the apparent weight of objects at the equator of the Earth to zero? [Answer: 1.4 hours]
(e) Astronomers have discovered objects they called pulsars, which emit bursts of radiation at regular intervals of less than a second. If a pulsar is to be interpreted as a rotating sphere beaming

out a natural "searchlight" that sweeps past the earth with each rotation, use your equation from c to show that its density would have to be much greater than that of ordinary matter.
(f) Astrophysicists predicted decades ago that certain stars that used up their sources of energy could collapse, forming a ball of neutrons with the fantastic density of $\sim 10^{17}$ kg/m^3. If this is what pulsars really are, use your equation from c to explain why no pulsar has ever been observed that flashes with a period of less than 1 ms or so.


Problem 11-j5.

11-j5 The figure shows an image from the Galileo space probe taken during its August 1993 flyby of the asteroid Ida. Astronomers were surprised when Galileo detected a smaller object orbiting Ida. This smaller object, the only known satellite of an asteroid in our solar system, was christened Dactyl, after the mythical creatures who lived on Mount Ida, and who protected the infant Zeus. For scale, Ida is about the size and shape of Orange County, and Dactyl the size of a college campus. Galileo was unfortunately unable to measure the time, T, required for Dactyl to orbit Ida. If it had, astronomers would have been able to make the first accurate determination of the mass and density of an asteroid. Find an equation for the density, ρ, of Ida in terms of Ida's known volume, V, the known radius, r, of Dactyl's orbit, and the lamentably unknown variable T. (This is the same technique that was used successfully for determining the masses and densities of the planets that have moons.)
▷ Solution, p. 203

11-j6 On an airless body such as the moon, there is no atmospheric friction, so it should be possible for a satellite to orbit at a very low altitude, just high enough to keep from hitting the mountains. (a) Suppose that such a body is a smooth sphere of uniform density ρ and radius r. Find the velocity required for a ground-skimming orbit. ✓
(b) A typical asteroid has a density of about 2 g/cm^3, i.e., twice that of water. (This is a lot lower than the density of the earth's crust, probably indicating that the low gravity is not enough to compact the material very tightly, leaving lots of empty space inside.) Suppose that it is possible for an astronaut in a spacesuit to jump at 2 m/s. Find the radius of the largest asteroid on which it would be possible to jump into a ground-skimming orbit.
✓

11-j7 If a bullet is shot straight up at a high enough velocity, it will never return to the earth. This is known as the escape velocity. In this problem, you will analyze the motion of an object of mass m whose initial velocity is *exactly* equal to escape velocity. We assume that it is starting from the surface of a spherically symmetric planet of mass M and radius b. The trick is to guess at the general form of the solution, and then determine the solution in more detail. Assume (as is true) that the solution is of the form $r = kt^p$, where r is the object's distance from the center of the planet at time t, and k and p are constants.
(a) Find the acceleration, and use Newton's second law and Newton's law of gravity to determine k and p. You should find that the result is independent of m. ✓
(b) What happens to the velocity as t approaches infinity?
(c) Determine escape velocity from the Earth's surface.
✓

11-j8 Planet X rotates, as the earth does, and is perfectly spherical. An astronaut who weighs 980.0 N on the earth steps on a scale at the north pole of Planet X and the scale reads 600.0 N; at the equator of Planet X, the scale only reads 500.0 N. The distance from the north pole to the equator is 20,000 km, measured along the surface of Planet X.
(a) Explain why the astronaut appears to weigh more at the north pole of planet X than at the equator. Which is the "actual" weight of the astronaut? Analyze the forces acting on an astronaut standing on a scale, providing one analysis for the north pole, and one for the equator.
(b) Find the mass of planet X. Is planet X more massive than the earth, or less massive? The radius of the earth is 6370 km, and its mass is 5.97×10^{24} kg. ✓
(c) If a 30,000 kg satellite is orbiting the planet very close to the surface, what is its orbital period? Assume planet X has no atmosphere, so that there's no air resistance. ✓
(d) How long is a day on planet X? Is this longer than, or shorter than, the period of the satellite in part c? ✓

11-j9 A 20.0 kg satellite has a circular orbit with a period of 2.40 hours and a radius of 8.00×10^6 m around planet Z. The magnitude of the gravitational acceleration on the surface of the planet is 8.00 m/s^2.
(a) What is the mass of planet Z? ✓
(b) What is the radius of planet Z? ✓
(c) Find the KE and the PE of the satellite. What is the ratio PE/KE (including both magnitude and sign)? You should get an integer. This is a special case of something called the *virial theorem*. ✓
(d) Someone standing on the surface of the planet sees a moon orbiting the planet (a circular orbit) with a period of 20.0 days. What is the distance between planet Z and its moon? ✓

11-m1 Astronomers calculating orbits of planets often work in a nonmetric system of units, in which the unit of time is the year, the unit of mass is the sun's mass, and the unit of distance is the astronomical unit (A.U.), defined as half the long axis of the earth's orbit. In these units, find an exact expression for the gravitational constant, G. ✓

11-m2 Suppose that we inhabited a universe in which, instead of Newton's law of gravity, we had $F = k\sqrt{m_1 m_2}/r^2$, where k is some constant with different units than G. (The force is still attractive.) However, we assume that $a = F/m$ and the rest of Newtonian physics remains true, and we use $a = F/m$ to define our mass scale, so that, e.g., a mass of 2 kg is one which exhibits half the acceleration when the same force is applied to it as to a 1 kg mass.
(a) Is this new law of gravity consistent with Newton's third law?
(b) Suppose you lived in such a universe, and you dropped two unequal masses side by side. What would happen?
(c) Numerically, suppose a 1.0-kg object falls with an acceleration of 10 m/s^2. What would be the acceleration of a rain drop with a mass of 0.1 g? Would you want to go out in the rain?
(d) If a falling object broke into two unequal pieces while it fell, what would happen?
(e) Invent a law of gravity that results in behavior that is the opposite of what you found in part b. [Based on a problem by Arnold Arons.]

11-m3 The structures that we see in the universe, such as solar systems, galaxies, and clusters of galaxies, are believed to have condensed from clumps that formed, due to gravitational attraction, in preexisting clouds of gas and dust. Observations of the cosmic microwave background radiation suggest that the mixture of hot hydrogen and helium that existed soon after the Big Bang was extremely uniform, but not perfectly so. We can imagine that any region that started out a little more dense would form a

natural center for the collapse of a clump. Suppose that we have a spherical region with density ρ and radius r, and for simplicity let's just assume that it's surrounded by vacuum. (a) Find the acceleration of the material at the edge of the cloud. To what power of r is it proportional? ✓

(b) The cloud will take a time t to collapse to some fraction of its original size. Show that t is independent of r.

Remark: This result suggests that structures would get a chance to form at all scales in the universe. That is, solar systems would not form before galaxies got to, or vice versa. It is therefore physically natural that when we look at the universe at essentially all scales less than a billion light-years, we see structure.

11-m4 You have a fixed amount of material with a fixed density. If the material is formed into some shape S, then there will be some point in space at which the resulting gravitational field attains its maximum value g_S. What shape maximizes g_S?

11-m5 The *escape velocity* of a massive body is the speed for which the total energy of a projectile is zero: the projectile has just enough KE to move infinitely far away from the massive body, with no left-over KE. The escape velocity depends on the distance from which the projectile is launched — often the body's surface.

The Schwarzschild radius (R_s) of a massive body is the radius where the escape velocity is equal to the speed of light, $c = 3.00 \times 10^8$ m/s. An object is called a *black hole* if it has a Schwarzschild radius.

An object must be very compact to be a black hole. For example, escape velocity from the surface of the earth is tens of thousands of times less than c, as is the escape velocity for a projectile launched from the center of the earth through a hypothetical radial, evacuated tunnel.

In this problem we will make some numerical estimates of how compact an object has to be in order to be a black hole. We will use Newtonian gravity, which is a poor approximation for such strong gravitational fields, so we expect these estimates to be rough.

(a) For an object of mass M, what would its radius have to be if all of its mass was to fit within the Schwarzschild radius? ✓

(b) Evaluate your equation from part a for M equal to the masses of the earth and the sun. If these bodies were compressed to approximately these sizes, they would become black holes. (Because these are rough estimates, treat them as having no more than 1 significant figure.) ✓

11-m6 Problems 11-m6-11-m8 all investigate the following idea. Cosmological surveys at the largest observable distance scales have detected structures like filaments. As an idealization of such a structure, consider a uniform mass distribution lying along the entire x axis, with mass density λ in units of kg/m. The purpose of this problem is to find the gravitational field created by this structure at a distance y.

(a) Determine as much as possible about the form of the solution, based on units.

(b) To evaluate the actual result, find the contribution $\mathrm{d}g_y$ to the y component of the field arising from the mass $\mathrm{d}m$ lying between x and $x + \mathrm{d}x$, then integrate it.

▷ Solution, p. 204

11-m7 Let us slightly change the physical situation described in problem 11-m6, letting the filament have a finite size, while retaining its symmetry under rotation about the x axis. The details don't actually matter very much for our purposes, but if we like, we can take the mass density to be constant within a cylinder of radius b centered on the x axis. Now consider the following two limits:

$$g_1 = \lim_{y\to 0} \lim_{b\to 0} g \qquad \text{and}$$

$$g_2 = \lim_{b\to 0} \lim_{y\to 0} g.$$

Each of these is a limit inside another limit, the only difference being the order of the limits. Either of these could be used as a definition of the

field at a point *on* an infinitely thin filament. Do they agree?

11-m8 Suppose we have a mass filament like the one described in problems 11-m6 and 11-m7, but now rather than taking it to be straight, let it have the shape of an arbitrary smooth curve. Locally, "under a microscope," this curve will look like an arc of a circle, i.e., we can describe its shape solely in terms of a radius of curvature. As in problem 11-m7, consider a point P lying *on* the filament itself, taking g to be defined as in definition g_1. Investigate whether g is finite, and also whether it points in a specific direction. To clarify the mathematical idea, consider the following two limits:

$$A = \lim_{x\to 0} \frac{1}{x} \qquad \text{and}$$
$$B = \lim_{x\to 0} \frac{1}{x^2}.$$

We say that $A = \infty$, while $B = +\infty$, i.e., both diverge, but B diverges with a definite sign. For a straight filament, as in problem 11-m6, with an infinite radius of curvature, symmetry guarantees that the field at P has no specific direction, in analogy with limit A. For a curved filament, a calculation is required in order to determine whether we get behavior A or B. Based on your result, what is the expected dynamical behavior of such a filament?

11-p1 (a) If the earth was of uniform density, would your weight be increased or decreased at the bottom of a mine shaft? Explain.
(b) In real life, objects weigh slightly more at the bottom of a mine shaft. What does that allow us to infer about the Earth?

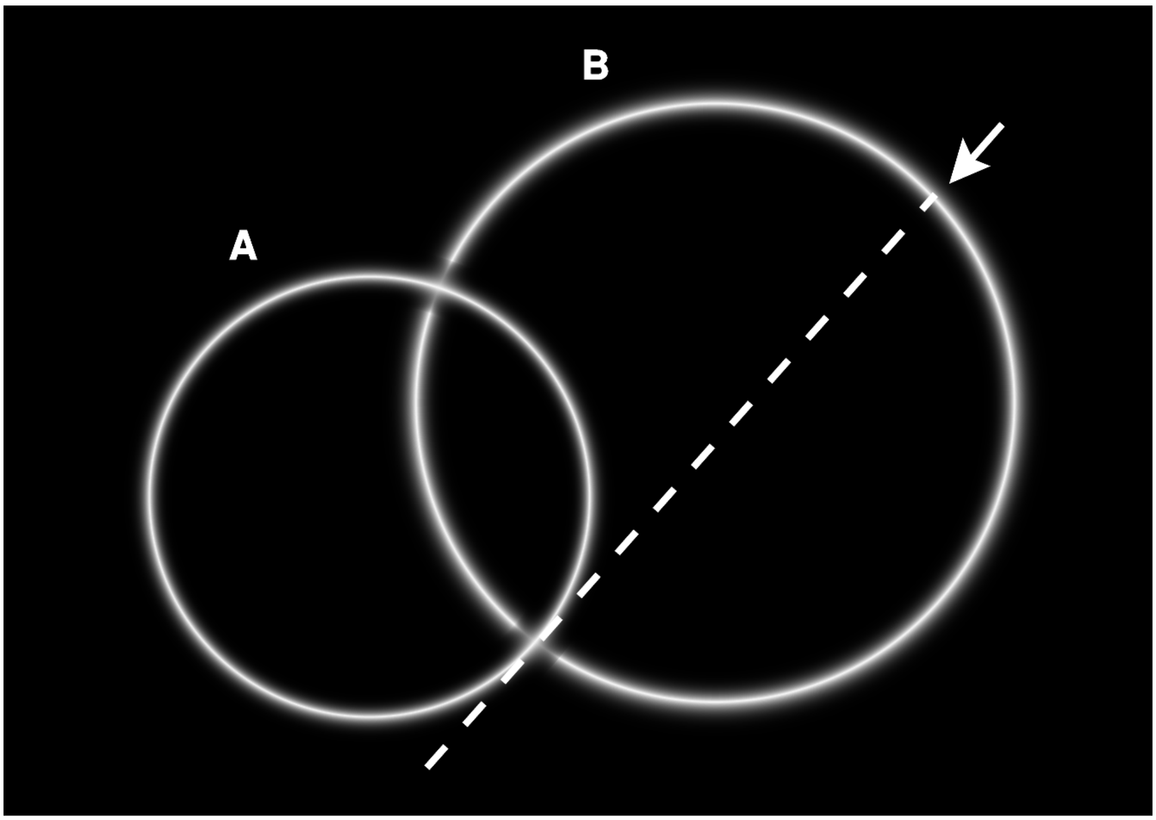
11-p2 Consult a proof of the shell theorem in your textbook or in some other source such as Wikipedia.Prove that the theorem fails if the exponent of r in Newton's law of gravity differs from -2.

11-p3 The shell theorem describes two cases, inside and outside. Show that for an alternative law of gravity $F = GMmr$ (with r^1 rather than r^{-2}), the outside case still holds.

11-p4 The figure shows a region of outer space in which two stars have exploded, leaving behind two overlapping spherical shells of gas, which we assume to remain at rest. The figure is a cross-section in a plane containing the shells' centers. A space probe is released with a very small initial speed at the point indicated by the arrow, initially moving in the direction indicated by the dashed line. Without any further information, predict as much as possible about the path followed by the probe and its changes in speed along that path.

11-p5 Approximate the earth's density as being constant. (a) Find the gravitational field at a point P inside the earth and half-way between the center and the surface. Express your result as a ratio g_P/g_S relative to the field we experience at the surface. (b) As a check on your answer, make sure that the same reasoning leads to a reasonable result when the fraction 1/2 is replaced by the value 0 (P being the earth's center) or the value 1 (P being a point on the surface).

11-p6 The earth is divided into solid inner core, a liquid outer core, and a plastic mantle. Physical properties such as density change discontinuously at the boundaries between one layer and the next. Although the density is not completely constant within each region, we will approximate it as being so for the purposes of this problem. (We neglect the crust as well.) Let R be the radius of the earth as a whole and M its mass. The following table gives a model of some properties of the three layers, as determined by methods such as the observation of earthquake waves that have propagated from one side of the planet to the other.



Problem 11-p4.

region	*outer radius/R*	*mass/M*
mantle	1	0.69
outer core	0.55	0.29
inner core	0.19	0.017

The boundary between the mantle and the outer core is referred to as the Gutenberg discontinuity. Let g_s be the strength of the earth's gravitational field at its surface and g_G its value at the Gutenberg discontinuity. Find g_G/g_s.

√

11-s1 Starting at a distance r from a planet of mass M, how fast must an object be moving in order to have a hyperbolic orbit, i.e., one that never comes back to the planet? This velocity is called the escape velocity. Interpreting the result, does it matter in what direction the velocity is? Does it matter what mass the object has? Does the object escape because it is moving too fast for gravity to act on it?

√

11-s2 A certain binary star system consists of two stars with masses m_1 and m_2, separated by a distance b. A comet, originally nearly at rest in deep space, drops into the system and at a certain point in time arrives at the midpoint between the two stars. For that moment in time, find its velocity, v, symbolically in terms of b, m_1, m_2, and fundamental constants.

√

12 Oscillations

This is not a textbook. It's a book of problems meant to be used along with a textbook. Although each chapter of this book starts with a brief summary of the relevant physics, that summary is not meant to be enough to allow the reader to actually learn the subject from scratch. The purpose of the summary is to show what material is needed in order to do the problems, and to show what terminology and notation are being used.

12.1 Periodic motion

The sine function has the property that $\sin(x+2\pi) = \sin x$, so that whatever the function is doing at one point, it is guaranteed to do the same thing again at a point 2π to the right. Such a function is called periodic. When an object's position as a function of time is periodic, we say that it exhibits periodic motion. Examples include uniform circular motion and a mass vibrating back and forth frictionlessly on a spring. The time from one repetition of the motion to the next is called the *period*, T. The inverse of the period is the *frequency*,

$$f = \frac{1}{T},$$

and it is also convenient to define the angular frequency

$$\omega = \frac{2\pi}{T}.$$

Either f or ω can be referred to simply as frequency, when context makes it clear or the distinction isn't important. The units of frequency are s^{-1}, which can also be abbreviated as hertz, $1\ \mathrm{Hz} = 1\ \mathrm{s}^{-1}$.

12.2 Simple harmonic motion

When an object is displaced from equilibrium, it can oscillate around the equilibrium point. Let the motion be one-dimensional, let the equilibrium be at $x = 0$, and let friction be negligible. Then by conservation of energy, the oscillations are periodic, and they extend from some negative value of x on the left to a value on the right that is the same except for the sign. We describe the size of the oscillations as their *amplitude*, A.

When the oscillations are small enough, Hooke's law $F = -kx$ is a good approximation, because any function looks linear close up. Therefore all such oscillations have a universal character, differing only in amplitude and frequency. Such oscillations are referred to as *simple harmonic motion.*

For simple harmonic motion, Newton's second law gives $x'' = -(k/m)x$. This is a type of equation referred to as a differential equation, because it relates the function $x(t)$ to its own (second) derivative. The solution of the equation is $x = A\sin(\omega t + \delta)$, where

$$\omega = \sqrt{\frac{k}{m}}$$

is independent of the amplitude.

12.3 Damped oscillations

The total energy of an oscillation is proportional to the square of the amplitude. In the simple harmonic oscillator, the amplitude and energy are constant. Unlike this idealization, real oscillating systems have mechanisms such as friction that dissipate energy. These mechanisms are referred to as damping. A simple mathematical model that incorporates this behavior is to incorporate a frictional force that is proportional to velocity. The equation of motion then becomes $mx'' + bx' + kx = 0$. In the most common case, where $b < 2\sqrt{km}$. In this *underdamped* case, the solutions are decaying exponentials of the form

$$x = Ae^{-ct}\sin\omega t,$$

where $c = b/2m$ and $\omega = [k/m - b^2/4m^2]^{1/2}$.

It is customary to describe the amount of damping with a quantity called the *quality factor*, Q, defined as the number of cycles required for the energy to fall off by a factor of $e^{2\pi} \approx 535$. The terminology arises from the fact that friction is often considered a bad thing, so a mechanical device that can vibrate for many oscillations before it loses a significant fraction of its energy would be considered a high-quality device.

Underdamped motion occurs for $Q > 1/2$. For the case $Q < 1/2$, referred to as *overdamped*, there are no oscillations, and the motion is a decaying exponential.

12.4 Driven oscillations

It is often of interest to consider an oscillator that is driven by an oscillating force. Examples would be a mother pushing a child on a playground swing, or the ear responding to a sound wave. We assume for simplicity that the driving force oscillates sinusoidally with time, although most of the same results are qualitatively correct when this requirement is relaxed. The oscillator responds to the driving force by gradually settling down into a steady, sinusoidal pattern of vibration called the steady state. Figure 12.1 shows the bell-shaped curve that results when we graph the energy of the steady-state response against the frequency of the driving force. We have the following results.

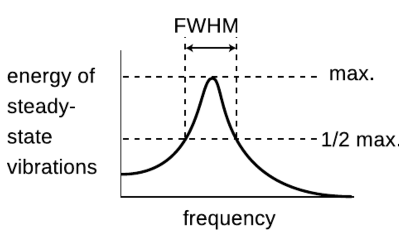


Figure 12.1: The response of an oscillator to a driving force, showing the definition of the full width at half maximum (FWHM).

(1) The steady-state response to a sinusoidal driving force occurs at the frequency of the force, not at the system's own natural frequency of vibration.

(2) A vibrating system *resonates* at its own natural frequency.[1] That is, the amplitude of the steady-state response is greatest in proportion to the amount of driving force when the driving force matches the natural frequency of vibration.

(3) When a system is driven at resonance, the steady-state vibrations have an amplitude that is proportional to Q.

(4) The FWHM of a resonance, defined in figure 12.1, is related to its Q and its resonant frequency f_{res} by the equation

$$\text{FWHM} = \frac{f_{res}}{Q}.$$

(This equation is only a good approximation when Q is large.)

[1] This is an approximation, which is valid in the usual case where Q is significantly greater than 1.

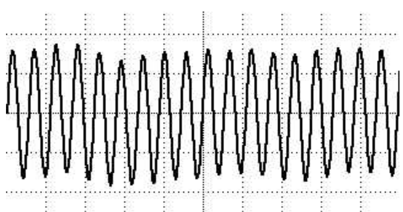
Problems

12-a1 Many single-celled organisms propel themselves through water with long tails, which they wiggle back and forth. (The most obvious example is the sperm cell.) The frequency of the tail's vibration is typically about 10-15 Hz. To what range of periods does this range of frequencies correspond?

$\checkmark$

12-a2 The figure shows the oscillation of a microphone in response to the author whistling the musical note "A." The horizontal axis, representing time, has a scale of 1.0 ms per square. Find the period T, the frequency f, and the angular frequency ω.

$\checkmark$



Problem 12-a2.

12-d1 Show that the equation $T = 2\pi\sqrt{m/k}$ has units that make sense.

12-d2 (a) Pendulum 2 has a string twice as long as pendulum 1. If we define x as the distance traveled by the bob along a circle away from the bottom, how does the k of pendulum 2 compare with the k of pendulum 1? Give a numerical ratio. [Hint: the total force on the bob is the same if the angles away from the bottom are the same, but equal angles do not correspond to equal values of x.]

(b) Based on your answer from part (a), how does the period of pendulum 2 compare with the period of pendulum 1? Give a numerical ratio.

12-d3 A pneumatic spring consists of a piston riding on top of the air in a cylinder. The upward force of the air on the piston is given by $F_{air} = ax^{-\beta}$, where $\beta = 1.4$ and a is a constant with funny units of $\mathrm{N \cdot m^{1.4}}$. For simplicity, assume the air only supports the weight mg of the piston itself, although in practice this device is used to support some other object. The equilibrium position, x_0, is where mg equals $-F_{air}$. (Note that in the main text I have assumed the equilibrium position to be at $x = 0$, but that is not the natural choice here.) Assume friction is negligible, and consider a case where the amplitude of the vibrations is very small. Find the angular frequency of oscillation.

$\checkmark$

12-d4 Verify that energy is conserved in simple harmonic motion.

12-d5 Archimedes' principle states that an object partly or wholly immersed in fluid experiences a buoyant force equal to the weight of the fluid it displaces. For instance, if a boat is floating in water, the upward pressure of the water (vector sum of all the forces of the water pressing inward and upward on every square inch of its hull) must be equal to the weight of the water displaced, because if the boat was instantly removed and the hole in the water filled back in, the force of the surrounding water would be just the right amount to hold up this new "chunk" of water. (a) Show that a cube of mass m with edges of length b floating upright (not tilted) in a fluid of density ρ will have a draft (depth to which it sinks below the waterline) h given at equilibrium by $h_0 = m/b^2\rho$. (b) Find the total force on the cube when its draft is h, and verify that plugging in $h - h_0$ gives a total force of zero. (c) Find the cube's period of oscillation as it bobs up and down in the water, and show that can be expressed in terms of and g only.

$\checkmark$

12-d6 A hot scientific question of the 18th century was the shape of the earth: whether its radius was greater at the equator than at the poles, or the other way around. One method used to attack this question was to measure gravity accurately in different locations on the

earth using pendula. If the highest and lowest latitudes accessible to explorers were 0 and 70 degrees, then the the strength of gravity would in reality be observed to vary over a range from about 9.780 to 9.826 m/s^2. This change, amounting to 0.046 m/s^2, is greater than the 0.022 m/s^2 effect to be expected if the earth had been spherical. The greater effect occurs because the equator feels a reduction due not just to the acceleration of the spinning earth out from under it, but also to the greater radius of the earth at the equator. What is the accuracy with which the period of a one-second pendulum would have to be measured in order to prove that the earth was not a sphere, and that it bulged at the equator?

$\surd$

12-d7 A certain mass, when hung from a certain spring, causes the spring to stretch by an amount h compared to its equilibrium length. If the mass is displaced vertically from this equilibrium, it will oscillate up and down with a period T_{osc}. Give a numerical comparison between T_{osc} and T_{fall}, the time required for the mass to fall from rest through a height h, when it isn't attached to the spring. (You will need the result of problem **??**).

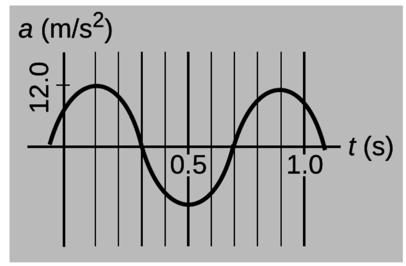
$\surd$

12-d8 An object undergoing simple harmonic motion oscillates with position $x(t) = (35\text{ cm})\cos[(25\text{ s}^{-1})t + \pi]$.
(a) Find the period, angular frequency, and frequency. $\surd$
(b) What is the initial velocity? What is the maximum speed? $\surd$
(c) What is the initial acceleration? What is the maximum in magnitude of the acceleration? $\surd$
(d) Find the location and velocity of the object at 1.00 s.

$\surd$

12-d9 On a frictionless, horizontal air track, a glider oscillates at the end of an ideal spring of force constant 150 N/m. The graph shows the acceleration of the glider as a function of time.
(a) Find the period of oscillations and the mass of the glider. $\surd$
(b) Notice that the graph says that the maximum acceleration of the glider is 12.0 m/s^2. Use this to find the amplitude of oscillations, A. $\surd$
(c) When the glider is $A/3$ away from its equilibrium position, what are its kinetic and potential energies? $\surd$
(d) Suppose $x(t) = A\cos(\omega t+\phi)$, where $0 \le \phi < 2\pi$. Find ϕ.

$\surd$

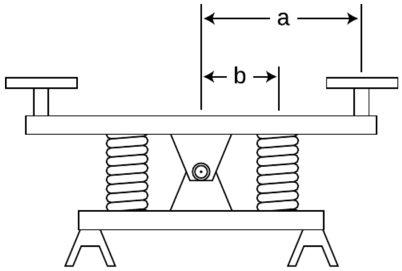


Problem 12-d9.

12-g1 Consider the same pneumatic piston described in problem 12-d3, but now imagine that the oscillations are not small. Sketch a graph of the total force on the piston as it would appear over this wider range of motion. For a wider range of motion, explain why the vibration of the piston about equilibrium is not simple harmonic motion, and sketch a graph of x vs t, showing roughly how the curve is different from a sine wave. [Hint: Acceleration corresponds to the curvature of the $x - t$ graph, so if the force is greater, the graph should curve around more quickly.]

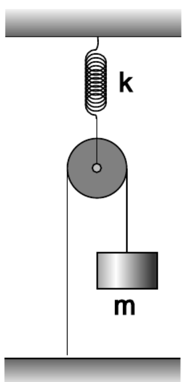
12-g2 The figure shows a see-saw with two springs at Codornices Park in Berkeley, California. Each spring has spring constant k, and a kid of mass m sits on each seat. (a) Find the period of vibration in terms of the variables k, m, a, and b. (b) Discuss the special case where $a = b$, rather than $a > b$ as in the real see-saw. (c) Show that your answer to part a also makes sense in the case of $b = 0$.

$\surd$



Problem 12-g2.

12-g3 Find the period of vertical oscillations of the mass m. The spring, pulley, and ropes have negligible mass. ✓

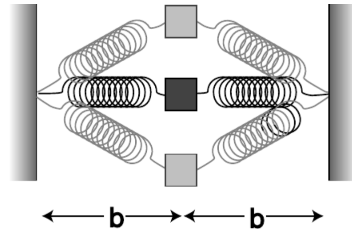


Problem 12-g3.

12-g4 The equilibrium length of each spring in the figure is b, so when the mass m is at the center, neither spring exerts any force on it. When the mass is displaced to the side, the springs stretch; their spring constants are both k.

(a) Find the energy, U, stored in the springs, as a function of y, the distance of the mass up or down from the center. ✓

(b) Show that the period of small up-down oscillations is infinite.



Problem 12-g4.

12-g5 For a one-dimensional harmonic oscillator, the solution to the energy conservation equation,

$$U + K = \frac{1}{2}kx^2 + \frac{1}{2}mv^2 = \text{constant},$$

is an oscillation with frequency $\omega = \sqrt{k/m}$. Now consider an analogous system consisting of a bar magnet hung from a thread, which acts like a magnetic compass. A normal compass is full of water, so its oscillations are strongly damped, but the magnet-on-a-thread compass has very little friction, and will oscillate repeatedly around its equilibrium direction. The magnetic energy of the bar magnet is

$$U = -Bm\cos\theta,$$

where B is a constant that measures the strength of the earth's magnetic field, m is a constant that parametrizes the strength of the magnet, and θ is the angle, measured in radians, between the bar magnet and magnetic north. The equilibrium occurs at $\theta = 0$, which is the minimum of U.

(a) By making analogies between rotational and linear motion, translate the equation defining the linear quantity k to one that defines an analogous angular one κ (Greek letter kappa). Applying this to the present example, find an expression for κ. (Assume the thread is so thin that its stiffness does not have any significant effect compared to earth's magnetic field.) ✓

(b) Find the frequency of the compass's vibrations. ✓

12-j1 A mass m on a spring oscillates around an equilibrium at $x = 0$. Any function $F(x)$ with an equilibrium at $x = 0$, $F(0) = 0$, can be approximated as $F(x) = -kx$, and if the spring's behavior is symmetric with respect to positive and negative values of x, so that $F(-x) = -F(x)$, then the next level of improvement in such an approximation would be $F(x) = -kx - bx^3$. The general idea here is that any smooth function can be approximated locally by a polynomial, and if you want a better approximation, you can use a polynomial with more terms in it. When you ask your calculator to calculate a function like sin or e^x, it's using a polynomial approximation with 10 or 12 terms. Physically, a spring with a positive value of b gets stiffer when stretched strongly than an "ideal" spring with $b = 0$. A spring with a negative b is like a person who cracks under stress — when you stretch it too much, it becomes more elastic than an ideal spring would. We should not expect any spring to give totally ideal behavior no matter no matter how much it is stretched. For example, there has to be some point at which it breaks.

Do a numerical simulation of the oscillation of a mass on a spring whose force has a nonvanishing b. Is the period still independent of amplitude? Is the amplitude-independent equation for the period still approximately valid for small enough amplitudes? Does the addition of an x^3 term with $b > 0$ tend to increase the period, or decrease it? Include a printout of your program and its output with your homework paper.

12-j2 An idealized pendulum consists of a pointlike mass m on the end of a massless, rigid rod of length L. Its amplitude, θ, is the angle the rod makes with the vertical when the pendulum is at the end of its swing. Write a numerical simulation to determine the period of the pendulum for any combination of m, L, and θ. Examine the effect of changing each variable while manipulating the others.

Problems 12-k1 through 12-k4 require specific knowledge of the properties of simple and physical pendulums.

12-k1 A simple pendulum of length L is released from from angle θ. Solve for the maximum speed of the pendulum bob two ways:
(a) Exactly, by using conservation of energy. $\surd$
(b) Approximately, by assuming $\theta \ll 1$, using $|v_{\max}| = A\omega$, and writing A and ω in terms of the given quantities. Your result is the first non-zero term in the Taylor expansion of the exact answer around $\theta = 0$.
$\surd$

12-k2 A pendulum of length L has period T on Earth. If a pendulum of length $2L$ has a period $4T$ on planet W, then what is the acceleration due to gravity on planet W? Give your answer to two significant figures.
$\surd$

12-k3 A uniform rod of length L is hung at one end. What is the period of oscillations for this physical pendulum?
$\surd$

12-k4 A pendulum with length L has period T when a very small mass is placed at the end of it (with size much less than L). Suppose we do not want to ignore the size of the bob. Consider a spherical bob with radius xL (x is a dimensionless constant, and L is the length of the string, connecting the pivot to the center of the bob). The period of motion of this physical pendulum is $T = 2\pi\sqrt{L/g}f(x)$. What is $f(x)$? Your expression for $f(x)$ should satisfy $f(0) = 1$. (Why?)
$\surd$

12-m1 If one stereo system is capable of producing 20 watts of sound power and another can put out 50 watts, how many times greater is the amplitude of the sound wave that can be created by the more powerful system? (Assume they are playing the same music.)

12-m2 What fraction of the total energy of an object undergoing SHM is kinetic at time $t = T/3$ (where T is the period of motion) if:
(a) the object is at maximum displacement from

equilibrium at $t = 0$; ✓
(b) the object is at equilibrium at $t = 0$. ✓

12-m3 An object undergoing simple harmonic motion has amplitude A and angular frequency ω. What is the speed of the object when it is at a distance $x = A/4$ from equilibrium? ✓

12-m4 A spring is attached to a wall as shown (the horizontal surface is frictionless). One fact is known about the spring: when compressed a distance 11.0 cm, the spring holds 1.00 J of elastic potential energy.
(a) What mass M must be attached to the spring so that it will oscillate with a period of 1.00 s? ✓
(b) If the amplitude of the motion is 5.00 cm and the period is that specified in part a, where is the object (relative to equilibrium) and in what direction is it moving 0.35 s after it has passed the equilibrium position, moving to the left? ✓
(c) At the instant described in part b, what are the kinetic and potential energies of the system? ✓
(d) What force (magnitude and direction) does the spring exert on mass M when it is 3.00 cm to the right of the equilibrium position, moving to the right?

12-p1 (a) Let W be the amount of work done by friction in the first cycle of oscillation, i.e., the amount of energy lost to heat. Find the fraction of the original energy E that remains in the oscillations after n cycles of motion.

(b) From this, prove the equation

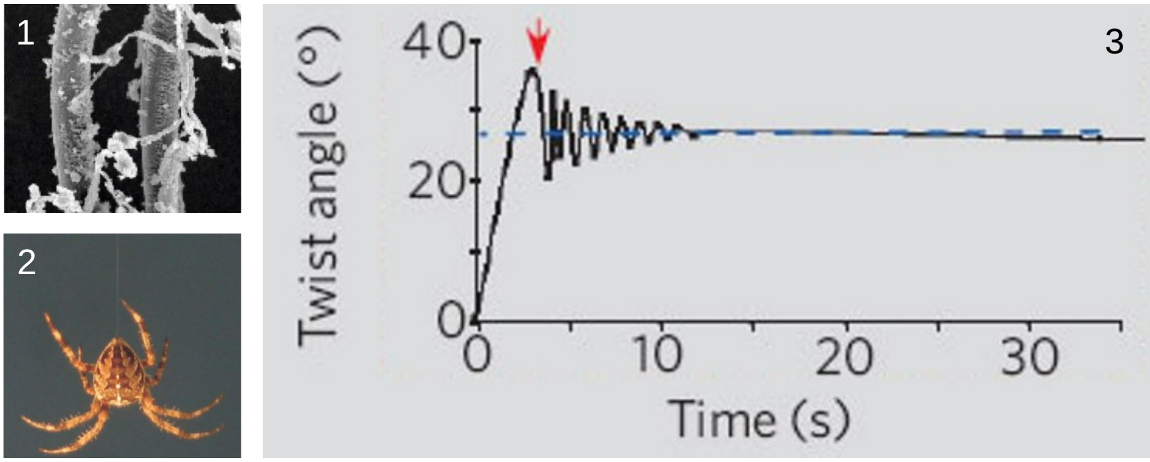
$$\left(1 - \frac{W}{E}\right)^Q = e^{-2\pi}$$

(recalling that the number 535 in the definition of Q is $e^{2\pi}$).

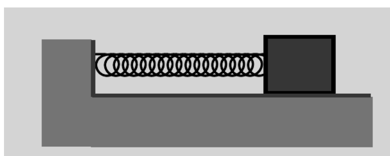
(c) Use this to prove the approximation $1/Q \approx (1/2\pi)W/E$. (Hint: Use the approximation $\ln(1+x) \approx x$, which is valid for small values of x, as shown on p. **??**.)

12-p2 (a) We observe that the amplitude of a certain free oscillation decreases from A_o to A_o/Z after n oscillations. Find its Q. ✓
(b) The figure is from *Shape memory in Spider draglines*, Emile, Le Floch, and Vollrath, *Nature* 440:621 (2006). Panel 1 shows an electron microscope's image of a thread of spider silk. In 2, a spider is hanging from such a thread. From an evolutionary point of view, it's probably a bad thing for the spider if it twists back and forth while hanging like this. (We're referring to a back-and-forth rotation about the axis of the thread, not a swinging motion like a pendulum.) The authors speculate that such a vibration could make the spider easier for predators to see, and it also seems to me that it would be a bad thing just because the spider wouldn't be able to control its orientation and do what it was trying to do. Panel 3 shows a graph of such an oscillation, which the authors measured using a video camera and a computer, with a 0.1 g mass hung from it in place of a spider. Compared to human-made fibers such as kevlar or copper wire, the spider thread has an unusual set of properties:

1. It has a low Q, so the vibrations damp out quickly.
2. It doesn't become brittle with repeated twisting as a copper wire would.
3. When twisted, it tends to settle in to a new equilibrium angle, rather than insisting on returning to its original angle. You can see this in panel 2, because although the experimenters initially twisted the wire by 35 degrees, the thread only performed oscillations with an amplitude much smaller than ± 35 degrees, settling down to a new equilibrium at 27 degrees.
4. Over much longer time scales (hours), the thread eventually resets itself to its original equlibrium angle (shown as zero degrees on the graph). (The graph reproduced here only shows the motion over a much shorter time scale.) Some human-made materials



Problem 12-p2.

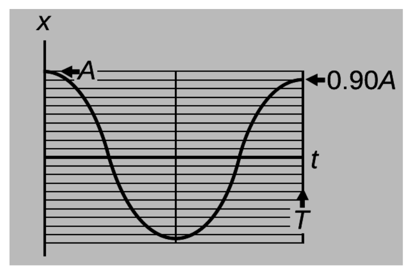


Problem 12-m4.

> have this "memory" property as well, but they typically need to be heated in order to make them go back to their original shapes.

Focusing on property number 1, estimate the Q of spider silk from the graph. $\checkmark$

12-p3 An object has underdamped motion as depicted in the figure, where $T = 2\pi/\omega$, and, as described in the text, ω differs from $\omega_0 = \sqrt{k/m}$.
(a) What fraction of the energy was lost during this first cycle? This fraction is lost in every ensuing cycle. $\checkmark$
(b) Where will the object be (in terms of A) after the second full oscillation? $\checkmark$
(c) By assuming $\omega \approx \omega_0$, what is the value of b? To express your answer, write $b = C\sqrt{km}$, and solve for the unitless constant C. $\checkmark$
(d) Use this value of b to find the percentage increase in the period of the motion as compared to the undamped case. You should get an answer much less than 1%, which means the approximation made in part c was justified. $\checkmark$ $\star$



Problem 12-p3.

12-s1 Many fish have an organ known as a swim bladder, an air-filled cavity whose main purpose is to control the fish's buoyancy and allow it to keep from rising or sinking without having to use its muscles. In some fish, however, the swim bladder (or a small extension of it) is linked to the ear and serves the additional purpose of amplifying sound waves. For a typical fish having such an anatomy, the bladder has a resonant frequency of 300 Hz, the bladder's Q is 3, and the maximum amplification is about a factor of 100 in energy. Over what range of frequencies

would the amplification be at least a factor of 50?

12-s2 As noted in section **??**, it is only approximately true that the amplitude has its maximum at the natural frequency $(1/2\pi)\sqrt{k/m}$. Being more careful, we should actually define two different symbols, $f_{\mathrm{o}} = (1/2\pi)\sqrt{k/m}$ and f_{res} for the slightly different frequency at which the amplitude is a maximum, i.e., the actual resonant frequency. In this notation, the amplitude as a function of frequency is

$$A = \frac{F}{2\pi\sqrt{4\pi^2 m^2\left(f^2 - f_0^2\right)^2 + b^2 f^2}}.$$

Show that the maximum occurs not at f_{o} but rather at

$$f_{\mathrm{res}} = \sqrt{f_0^2 - \frac{b^2}{8\pi^2 m^2}} = \sqrt{f_0^2 - \frac{1}{2}\mathrm{FWHM}^2}$$

Hint: Finding the frequency that minimizes the quantity inside the square root is equivalent to, but much easier than, finding the frequency that maximizes the amplitude.

12-s3 An oscillator with sufficiently strong damping has its maximum response at $\omega = 0$. Using the result derived on page **??** , find the value of Q at which this behavior sets in.

12-s4 The goal of this problem is to refine the proportionality FWHM $\propto f_{res}/Q$ into the equation FWHM $= f_{res}/Q$, i.e., to prove that the constant of proportionality equals 1.

(a) Show that the work done by a damping force $F = -bv$ over one cycle of steady-state motion equals $W_{damp} = -2\pi^2 bfA^2$. Hint: It is less confusing to calculate the work done over half a cycle, from $x = -A$ to $x = +A$, and then double it.

(b) Show that the fraction of the undriven oscillator's energy lost to damping over one cycle is $|W_{damp}|/E = 4\pi^2 bf/k$.

(c) Use the previous result, combined with the result of problem 4, to prove that Q equals $k/2\pi bf$.

(d) Combine the preceding result for Q with the equation FWHM $= b/2\pi m$ from section **??** to prove the equation FWHM $= f_{res}/Q$.

12-s5 An oscillator has Q=6.00, and, for convenience, let's assume $F_m = 1.00$, $\omega_{\mathrm{o}} = 1.00$, and $m = 1.00$. The usual approximations would give

$$\omega_{res} = \omega_{\mathrm{o}},$$
$$A_{res} = 6.00, \qquad \text{and}$$
$$\Delta\omega = 1/6.00.$$

Determine these three quantities numerically using the result derived on page **??** , and compare with the approximations.

13 Electrical interactions

This is not a textbook. It's a book of problems meant to be used along with a textbook. Although each chapter of this book starts with a brief summary of the relevant physics, that summary is not meant to be enough to allow the reader to actually learn the subject from scratch. The purpose of the summary is to show what material is needed in order to do the problems, and to show what terminology and notation are being used.

13.1 Charge and Coulomb's law

It appears superficially that nature has many different types of forces, such as frictional forces, normal forces, sticky forces, the force that lets bugs walk on water, the force that makes gunpowder explode, and the force that causes honey to flow so slowly. Actually, all of the forces on this list are manifestations of electrical interactions at the atomic scale. Like gravity, electricity is a $1/r^2$ force. The electrical counterpart of Newton's law of gravity is Coulomb's law,

$$F = \frac{k|q_1||q_2|}{r^2}, \tag{13.1}$$

where F is the magnitude of the force, k is a universal constant, r is the distance between the two interacting objects, and q_1 and q_2 are properties of the objects called their electric charges. This equation is often written using the alternate form of the constant $\epsilon_0 = 1/(4\pi k)$.

Electric charge is measured in units of Coulombs, C. Charge is to electricity as mass is to gravity. There are two types of charge, which are conventionally labeled positive and negative. Charges of the same type repel one another, and opposite charges attract. Charge is conserved.

Charge is quantized, meaning that all charges are integer multiples of a certain fundamental charge e. (The quarks that compose neutrons and protons have charges that come in thirds of this unit, but quarks are never observed individually, only in clusters that have integer multiples of e.) The electron has charge $-e$, the proton $+e$.

13.2 The electric field

Newton conceived of forces as acting instantaneously at a distance. We now know that if masses or charges in a certain location are moved around, the change in the force felt by a distant mass or charge is delayed. The effect travels at the speed of light, which according to Einstein's theory of relativity represents a maximum speed at which cause and effect can propagate, built in to the very structure of space and time. Because time doesn't appear as a variable in Coulomb's law, Coulomb's law cannot fundamentally be a correct description of electrical interactions. It is only an approximation, which is valid when charges are not moving (the science of electrostatics) or when the time lags in the propagation of electrical interactions are negligible. These considerations imply logically that while an electrical effect is traveling through space, it has its own independent physical reality. We think of space as being permeated with an electric field, which varies dynamically according to its own rules, even if there are no charges nearby. Phenomena such as visible light and radio waves are ripples in the electric (and magnetic) fields. For now we will study only static electric fields (ones that don't change with time), but fields come into their own when their own dynamics are important.

The electric field $\mathbf{E}$ at a given point in space can be defined in terms of the electric force $\mathbf{F}$ that would be exerted on a hypothetical test charge q_t inserted at that point:

$$\mathbf{E} = \frac{\mathbf{F}}{q_t}. \tag{13.2}$$

By a test charge, we mean one that is small enough so that its presence doesn't disturb the

situation that we're trying to measure. From the definition, we see that the electric field is a vector with units of newtons per coulumb, N/C. Its gravitational counterpart is the familiar **g**, whose magnitude on earth is about 9.8 m/s^2. Because forces combine according to the rules of vector addition, it follows that the electric field of a combination of charges is the vector sum of the fields that would have been produced individually by those charges.

The electric field contains energy. The electrical energy contained in an infinitesimal volume $\mathrm{d}v$ is given by $\mathrm{d}U_e = (1/8\pi k)E^2\,\mathrm{d}v$.

13.3 Conductors and insulators

Some materials, such as metals, are good electrical conductors, meaning that they contain charges that are free to move. A material like dry wood is an insulator, because it contains few such free charges. When a perfect conductor is in static equilibrium, any net charge is on the surface, and the electric field is zero on its interior. The electric field at the surface is perpendicular to the surface.

13.4 The electric dipole

When an unbalanced distribution of charges is subject to an external electric field **E**, it experiences a torque $\boldsymbol{\tau}$. We define the electric dipole moment **d** to be the vector such that

$$\boldsymbol{\tau} = \mathbf{d} \times \mathbf{E}. \tag{13.3}$$

When the total charge is zero, this relation uniquely defines **d**, regardless of the point chosen as the axis. In the simplest case, of charges $+q$ and $-q$ at opposite ends of a stick of length ℓ, the dipole moment has magnitude $q\ell$ and points from the negative charge to the positive one. The potential energy of a dipole in an external field is

$$U = -\mathbf{d} \cdot \mathbf{E}. \tag{13.4}$$

13.5 The field of a continuous charge distribution

The field of a continuous charge distribution can be found by integrating the contribution to the field from each infinitesimal part of the distribution.

13.6 Gauss's law

When we look at the "sea of arrows" representation of a field, 13.1/1, there is a natural visual tendency to imagine connecting the arrows as in 13.1/2. The curves formed in this way are called field lines, and they have a direction, shown by the arrowheads.

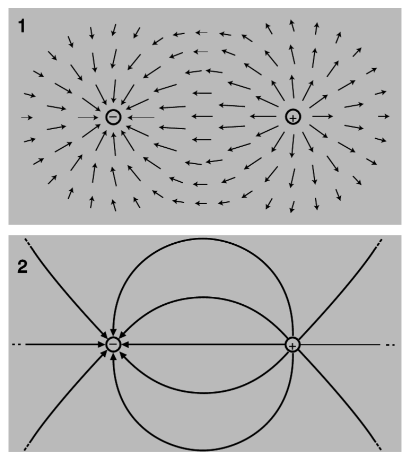


Figure 13.1: Two different representations of an electric field.

Electric field lines originate from positive charges and terminate on negative ones. We can choose a constant of proportionality that fixes how coarse or fine the "grain of the wood" is, but once this choice is made the strength of each charge is shown by the number of lines that begin or end on it. For example, figure 13.1/2 shows eight lines at each charge, so we know that $q_1/q_2 = (-8)/8 = -1$. Because lines never begin

or end except on a charge, we can always find the total charge inside any given region by subtracting the number of lines that go in from the number that come out and multiplying by the appropriate constant of proportionality. Ignoring the constant, we can apply this technique to figure 13.2 to find $q_A = -8$, $q_B = 2 - 2 = 0$, and $q_C = 5 - 5 = 0$.

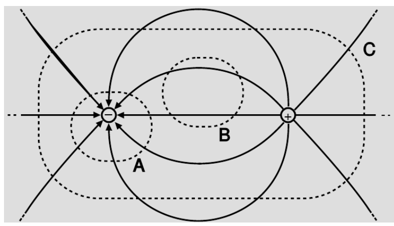


Figure 13.2: The number of field lines coming in and out of each region depends on the total charge it encloses.

Let us now make this description more mathematically precise. Given a smooth, closed surface such as the ones in figure 13.2, we have an inside and an outside, so that at any point on the surface we can define a unit normal $\hat{\mathbf{n}}$ (i.e., a vector with magnitude 1, perpendicular to the surface) that points outward. Given an infinitesimally small piece of the surface, with area $\mathrm{d}A$, we define an area vector $\mathrm{d}\mathbf{A} = \hat{\mathbf{n}}\,\mathrm{d}A$. The infinitesimal flux $\mathrm{d}\Phi$ through this infinitesimal patch of the surface is defined as $\mathrm{d}\Phi = \mathbf{E}\cdot\mathrm{d}\mathbf{A}$, and integrating over the entire surface gives the total flux $\Phi = \int \mathrm{d}\Phi = \int \mathbf{E}\cdot\mathrm{d}\mathbf{A}$. Intuitively, the flux measures how many field lines pierce the surface. Gauss's law states that

$$q_{\text{in}} = \frac{\Phi}{4\pi k}, \tag{13.5}$$

where q_{in} is the total charge inside a closed surface, and Φ is the flux through the surface. (In terms of the constant $\epsilon_0 = 1/(4\pi k)$, we have $q_{\text{in}} = \epsilon_0\Phi$.)

Unlike Coulomb's law, Gauss's law holds in all circumstances, even when there are charges moving in complicated ways and electromagnetic waves flying around. Gauss's law can be thought of as a definition of electric charge.

13.7 Gauss's law in differential form

Gauss' law is a bit spooky. It relates the field on the Gaussian surface to the charges inside the surface. What if the charges have been moving around, and the field at the surface right now is the one that was created by the charges in their previous locations? Gauss' law — unlike Coulomb's law — still works in cases like these, but it's far from obvious how the flux and the charges can still stay in agreement if the charges have been moving around.

For this reason, it would be more physically attractive to restate Gauss' law in a different form, so that it related the behavior of the field at one point to the charges that were actually present at that point. We define the *divergence* of the electric field,

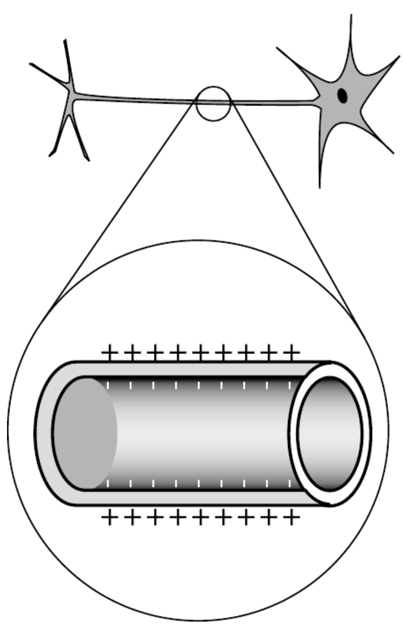
$$\operatorname{div}\mathbf{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}.$$

Gauss's law in differential form is

$$\operatorname{div}\mathbf{E} = 4\pi k\rho.$$

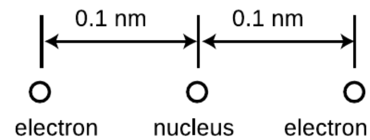
Problems

13-a1 The figure shows a neuron, which is the type of cell your nerves are made of. Neurons serve to transmit sensory information to the brain, and commands from the brain to the muscles. All this data is transmitted electrically, but even when the cell is resting and not transmitting any information, there is a layer of negative electrical charge on the inside of the cell membrane, and a layer of positive charge just outside it. This charge is in the form of various ions dissolved in the interior and exterior fluids. Why would the negative charge remain plastered against the inside surface of the membrane, and likewise why doesn't the positive charge wander away from the outside surface?



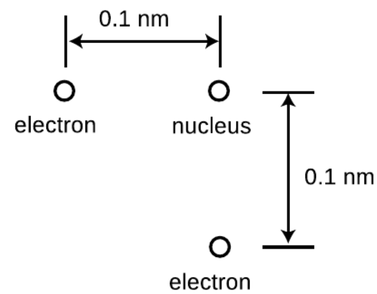
Problem 13-a1.

13-a2 A helium atom finds itself momentarily in this arrangement. Find the direction and magnitude of the force acting on the right-hand electron. The two protons in the nucleus are so close together ($\sim$ 1 fm) that you can consider them as being right on top of each other. ✓



Problem 13-a2.

13-a3 The helium atom of problem 13-a2 has some new experiences, goes through some life changes, and later on finds itself in the configuration shown here. What are the direction and magnitude of the force acting on the bottom electron? (Draw a sketch to make clear the definition you are using for the angle that gives direction.) ✓



Problem 13-a3.

13-a4 Suppose you are holding your hands in front of you, 10 cm apart.

(a) Estimate the total number of electrons in each hand. ✓

(b) Estimate the total repulsive force of all the electrons in one hand on all the electrons in the other. ✓

(c) Why don't you feel your hands repelling each other?

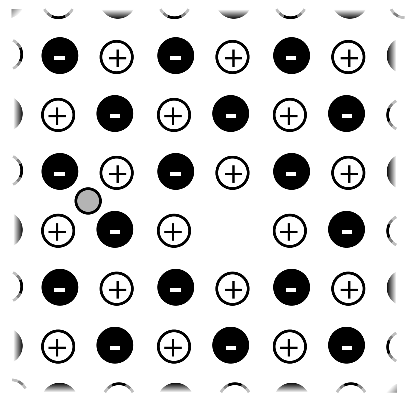
(d) Estimate how much the charge of a proton could differ in magnitude from the charge of an electron without creating a noticeable force between your hands.

13-a5 Suppose that a proton in a lead nucleus wanders out to the surface of the nucleus, and experiences a strong nuclear force of about 8 kN from the nearby neutrons and protons pulling it back in. Compare this numerically to the repulsive electrical force from the other protons, and verify that the net force is attractive. A lead nucleus is very nearly spherical, is about 6.5 fm in radius, and contains 82 protons, each with a charge of $+e$, where $e = 1.60 \times 10^{-19}$ C.
√

13-a6 The subatomic particles called muons behave exactly like electrons, except that a muon's mass is greater by a factor of 206.77. Muons are continually bombarding the Earth as part of the stream of particles from space known as cosmic rays. When a muon strikes an atom, it can displace one of its electrons. If the atom happens to be a hydrogen atom, then the muon takes up an orbit that is on the average 206.77 times closer to the proton than the orbit of the ejected electron. How many times greater is the electric force experienced by the muon than that previously felt by the electron?
√

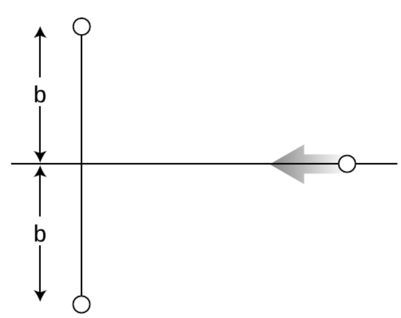
13-a7 The Earth and Moon are bound together by gravity. If, instead, the force of attraction were the result of each having a charge of the same magnitude but opposite in sign, find the quantity of charge that would have to be placed on each to produce the required force.
√

13-a8 The figure shows one layer of the three-dimensional structure of a salt crystal. The atoms extend much farther off in all directions, but only a six-by-six square is shown here. The larger circles are the chlorine ions, which have charges of $-e$, where $e = 1.60 \times 10^{-19}$ C. The smaller circles are sodium ions, with charges of $+e$. The center-to-center distance between neighboring ions is about 0.3 nm. Real crystals are never perfect, and the crystal shown here has two defects: a missing atom at one location, and an extra lithium atom, shown as a grey circle, inserted in one of the small gaps. If the lithium atom has a charge of $+e$, what is the direction and magnitude of the total force on it? Assume there are no other defects nearby in the crystal besides the two shown here.
√



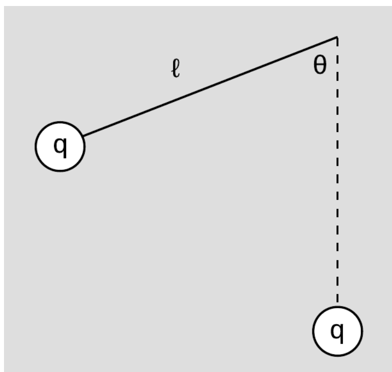
Problem 13-a8.

13-a9 In the semifinals of an electrostatic croquet tournament, Jessica hits her positively charged ball, sending it across the playing field, rolling to the left along the x axis. It is repelled by two other positive charges. These two equal charges are fixed on the y axis at the locations shown in the figure. (a) Express the force on the ball in terms of the ball's position, x. (b) At what value of x does the ball experience the greatest deceleration? Express you answer in terms of b. [Based on a problem by Halliday and Resnick.]
√



Problem 13-a9.

13-a10 As shown in the figure, a particle of mass m and charge q hangs from a string of length ℓ, forming a pendulum fixed at a central point. Another charge q is fixed at the same distance ℓ, directly below the center. Find the equilibrium values of θ and determine whether they are stable or unstable. ⋆



Problem 13-a10.

13-d1 (a) At time $t = 0$, a positively charged particle is placed, at rest, in a vacuum, in which there is a uniform electric field of magnitude E. Write an equation giving the particle's speed, v, in terms of t, E, and its mass and charge m and q. √

(b) If this is done with two different objects and they are observed to have the same motion, what can you conclude about their masses and charges? (For instance, when radioactivity was discovered, it was found that one form of it had the same motion as an electron in this type of experiment.)

13-d2 Three charges are arranged on a square as shown. All three charges are positive. What value of q_2/q_1 will produce zero electric field at the center of the square? √

13-d3 In an electrical storm, the cloud and the ground act like a parallel-plate capacitor, which typically charges up due to frictional electricity in collisions of ice particles in the cold upper atmosphere. Lightning occurs when the magnitude of the electric field builds up to a critical value, E_c, at which air is ionized.

(a) Treat the cloud as a flat square with sides of length L. If it is at a height h above the ground, find the amount of energy released in the lightning strike. √

(b) Based on your answer from part a, which is more dangerous, a lightning strike from a high-altitude cloud or a low-altitude one?

(c) Make an order-of-magnitude estimate of the energy released by a typical lightning bolt, assuming reasonable values for its size and altitude. E_c is about 10^6 N/C.

See problem **??** for a note on how recent research affects this estimate.

Problem 13-d3.

13-d4 The figure shows cross-sectional views of two cubical capacitors, and a cross-sectional view of the same two capacitors put together so that their interiors coincide. A capacitor with the plates close together has a nearly uniform electric field between the plates, and almost zero field outside; these capacitors don't have their plates very close together compared to the dimensions of the plates, but for the purposes of this problem, assume that they still have approximately the kind of idealized field pattern shown in the figure. Each capacitor has an interior volume of 1.00 m^3, and is charged up to the point where its internal field is 1.00 V/m.

(a) Calculate the energy stored in the electric field of each capacitor when they are separate.

√

(b) Calculate the magnitude of the interior field when the two capacitors are put together in the manner shown. Ignore effects arising from the redistribution of each capacitor's charge under the influence of the other capacitor. √

(c) Calculate the energy of the put-together configuration. Does assembling them like this release energy, consume energy, or neither?

√

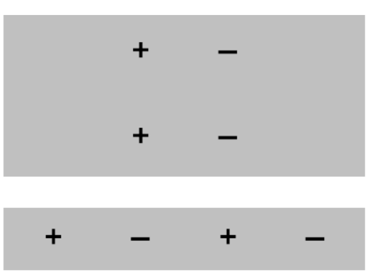
13-g1 The definition of the dipole moment, $\mathbf{D} = \sum q_i \mathbf{r}_i$, involves the vector $\mathbf{r}_i$ stretching from the origin of our coordinate system out to the charge q_i. There are clearly cases where this causes the dipole moment to be dependent on the choice of coordinate system. For instance, if there is only one charge, then we could make the dipole moment equal zero if we chose the origin to be right on top of the charge, or nonzero if we put the origin somewhere else.

(a) Make up a numerical example with two charges of equal magnitude and opposite sign. Compute the dipole moment using two different coordinate systems that are oriented the same way, but differ in the choice of origin. Comment on the result.

(b) Generalize the result of part a to any pair of charges with equal magnitude and opposite sign. This is supposed to be a proof for *any* arrangement of the two charges, so don't assume any numbers.

(c) Generalize further, to n charges.

13-g2 Compare the two dipole moments.



Problem 13-g2.

13-g3 Find an arrangement of charges that has zero total charge and zero dipole moment, but that will make nonvanishing electric fields.

13-g4 This is a one-dimensional problem, with everything confined to the x axis. Dipole A consists of a -1.000 C charge at $x = 0.000$ m and a 1.000 C charge at $x = 1.000$ m. Dipole B has a -2.000 C charge at $x = 0.000$ m and a 2.000 C charge at $x = 0.500$ m.

(a) Compare the two dipole moments.

(b) Calculate the field created by dipole A at $x = 10.000$ m, and compare with the field dipole B would make. Comment on the result.

√

13-g5 A dipole has a midplane, i.e., the plane that cuts through the dipole's center, and is perpendicular to the dipole's axis. Consider a two-charge dipole made of point charges $\pm q$ located at $z = \pm \ell/2$. Use approximations to find the field at a distant point in the midplane, and show that its magnitude comes out to be kD/R^3 (half what it would be at a point on the axis lying an equal distance from the dipole).

13-j1 Astronomers believe that the mass distribution (mass per unit volume) of some galaxies may be approximated, in spherical coordinates, by $\rho = ae^{-br}$, for $0 \le r \le \infty$, where ρ is the density. Find the total mass.

13-j2 A hydrogen atom in a particular state has the charge density (charge per unit volume) of the electron cloud given by $\rho = ae^{-br}z^2$, where r is the distance from the proton, and z is the coordinate measured along the z axis. Given that the total charge of the electron cloud must be $-e$, find a in terms of the other variables.

13-j3 (a) A rod of length L is uniformly charged with charge Q. It can be shown by integration that the field at a point lying in the midplane of the rod at a distance R is $E = k\lambda L / \left[R^2\sqrt{1 + L^2/4R^2}\right]$, where λ is the charge per unit length. Starting from this result, take the limit as the length of the rod

approaches infinity. Note that λ is not changing, so as L gets bigger, the total charge Q increases.

(b) It can be shown that the field of an infinite, uniformly charged plane is $2\pi k\sigma$. Now you're going to rederive the same result by a different method. Suppose that it is the $x-y$ plane that is charged, and we want to find the field at the point $(0,0,z)$. (Since the plane is infinite, there is no loss of generality in assuming $x=0$ and $y=0$.) Imagine that we slice the plane into an infinite number of straight strips parallel to the y axis. Each strip has infinitesimal width $\mathrm{d}x$, and extends from x to $x+\mathrm{d}x$. The contribution any one of these strips to the field at our point has a magnitude which can be found from part a. By vector addition, prove the desired result for the field of the plane of charge.

13-j4 Consider the electric field created by a uniformly charged cylindrical surface that extends to infinity in one direction.
(a) Show that the field at the center of the cylinder's mouth is $2\pi k\sigma$, which happens to be the same as the field of an infinite *flat* sheet of charge!
(b) This expression is independent of the radius of the cylinder. Explain why this should be so. For example, what would happen if you doubled the cylinder's radius?

13-j5 (a) Show that the energy in the electric field of a point charge is infinite! Does the integral diverge at small distances, at large distances, or both?
(b) Now calculate the energy in the electric field of a uniformly charged sphere with radius b. Based on the shell theorem, it can be shown that the field for $r>b$ is the same as for a point charge, while the field for $r<b$ is kqr/b^3. (Example **??** shows this using a different technique.)

Remark: The calculation in part a seems to show that infinite energy would be required in order to create a charged, pointlike particle. However, there are processes that, for example, create electron-positron pairs, and these processes don't require infinite energy. According to Einstein's famous equation $E=mc^2$, the energy required to create such a pair should only be $2mc^2$, which is finite. One way out of this difficulty is to assume that no particle is really pointlike, and this is in fact the main motivation behind a speculative physical theory called string theory, which posits that charged particles are actually tiny loops, not points.

$\checkmark$

13-j6 (a) A rod of length L is uniformly charged with charge Q. It can be shown by integration that the field at a point lying in the midplane of the rod at a distance R is $E=k\lambda L/\left[R^2\sqrt{1+L^2/4R^2}\right]$, where λ is the charge per unit length. Show that this field reduces to $E=2k\lambda/R$ in the limit of $L\rightarrow\infty$.
(b) An infinite strip of width b has a surface charge density σ. Find the field at a point at a distance z from the strip, lying in the plane perpendicularly bisecting the strip. $\checkmark$
(c) Show that this expression has the correct behavior in the limit where z approaches zero, and also in the limit of $z\gg b$. For the latter, you'll need the result of problem 13-j3a, which is given on page **??**.

13-j7 A solid cylinder of radius b and length ℓ is uniformly charged with a total charge Q. Find the electric field at a point at the center of one of the flat ends.

13-m1 A sphere of radius b contains a uniform charge density ρ. Use Gauss's law to find the electric field at radius $r\le b$, and verify that the result is the same as the one obtained using Newton's shell theorem for gravity. Problem by P. Widmann.

$\checkmark$

13-m2 A spherical shell of uniform charge density ρ extends from $r=a$ to $r=b$. Find the field in the regions $r\le a$, $a\le r\le b$, and $r\ge b$. Problem by P. Widmann.

$\checkmark$

13-m3 A hollow, conducting spherical shell, with zero total charge, has inner radius a and outer radius b. A point charge $+q$ is located at the center. Find the charges on the inner and

outer surfaces of the shell. Problem by P. Widmann.

$\checkmark$

13-m4 An infinite, uniform slab of charge with density ρ extends from $x = 0$ to $x = h$. The distant field on the left is zero. Find the field in the three regions $x \le 0$, $0 \le x \le h$, and $h \le x$. Problem by P. Widmann.

$\checkmark$

13-m5 Assume the earth is an infinite flat sheet, as some persons claim. Flat-earth cosmologies often omit any description of what's on the flip side, so let's assume that the gravitational field is zero there. If the density of the earth is $4\ \mathrm{g/cm^3}$, find the thickness that the earth must have in order to give $g = 9.8\ \mathrm{m/s^2}$ on our side. Problem by P. Widmann.

$\checkmark$

13-m6 Use Gauss' law to find the field inside an infinite cylinder with radius b and uniform charge density ρ.

$\checkmark$

13-m7 In a certain region of space, the electric field is constant (i.e., the vector always has the same magnitude and direction). For simplicity, assume that the field points in the positive x direction. (a) Use Gauss's law to prove that there is no charge in this region of space. This is most easily done by considering a Gaussian surface consisting of a rectangular box, whose edges are parallel to the x, y, and z axes.
(b) If there are no charges in this region of space, what could be making this electric field?

13-m8 (a) In a certain region of space, the electric field is given by $\mathbf{E} = bx\hat{\mathbf{x}}$, where b is a constant. Find the amount of charge contained within a cubical volume extending from $x = 0$ to $x = a$, from $y = 0$ to $y = a$, and from $z = 0$ to $z = a$.
(b) Repeat for $\mathbf{E} = bx\hat{\mathbf{z}}$.
(c) Repeat for $\mathbf{E} = 13bz\hat{\mathbf{z}} - 7cz\hat{\mathbf{y}}$.
(d) Repeat for $\mathbf{E} = bxz\hat{\mathbf{z}}$.

13-m9 Light is a wave made of electric and magnetic fields, and the fields are perpendicular to the direction of the wave's motion, i.e., they're transverse. An example would be the electric field given by $\mathbf{E} = b\hat{\mathbf{x}} \sin cz$, where b and c are constants. (There would also be an associated magnetic field.) We observe that light can travel through a vacuum, so we expect that this wave pattern is consistent with the nonexistence of any charge in the space it's currently occupying. Use Gauss's law to prove that this is true.

13-m10 An electric field is given in cylindrical coordinates (R, ϕ, z) by $E_R = ce^{-u|z|}R^{-1}\cos^2\phi$, where the notation E_R indicates the component of the field pointing directly away from the axis, and the components in the other directions are zero. (This isn't a completely impossible expression for the field near a radio transmitting antenna.) (a) Find the total charge enclosed within the infinitely long cylinder extending from the axis out to $R = b$. (b) Interpret the R-dependence of your answer to part a.

13-m11 An electron in an atom acts like a probability cloud surrounding the nucleus. For a hydrogen atom in its lowest-energy state, the probability falls off exponentially, so we can mock this up with a charge density $\rho = \rho_0 e^{-r/a}$, where r is the distance from the nucleus, and ρ_0 and a are constants. Find the electric field. Problem by P. Widmann.

$\checkmark$

14 The electric potential

This is not a textbook. It's a book of problems meant to be used along with a textbook. Although each chapter of this book starts with a brief summary of the relevant physics, that summary is not meant to be enough to allow the reader to actually learn the subject from scratch. The purpose of the summary is to show what material is needed in order to do the problems, and to show what terminology and notation are being used.

14.1 The electric potential

When a test charge q is at a particular position in a static electric field, it has an electrical potential energy U. The electrical potential energy per unit charge, U/q, is called the electric potential, notated V, φ, or Φ, and measured in units of volts, V. Because it is defined in terms of a potential energy, the electric potential is only defined up to an additive constant. In the context of an electric circuit, we often use the synonym "voltage" for the electric potential. Voltage differences are measured by a voltmeter, which reads the difference in potential between its two probes. A voltmeter is wired in parallel with a circuit, and an ideal voltmeter acts like a perfect insulator, so that no charge ever flows through it.

In one dimension, the electric field and electric potential are related by

$$E = -\frac{\mathrm{d}V}{\mathrm{d}x},$$

or equivalently, via the fundamental theorem of calculus,

$$V(x_2) - V(x_1) = -\int_{x_1}^{x_2} E\,\mathrm{d}x.$$

Generalizing to three dimensions,

$$\mathbf{E} = -\nabla V$$

(involving the gradient operator ∇) and

$$\Delta V = -\int \mathbf{E} \cdot \mathrm{d}\mathbf{x}$$

(in terms of a path integral). In electrostatics, the path integral in the latter equation is independent of the path taken.

Since the field of a charge distribution depends additively upon the charges, the same is true of the potential. Given a continuous charge distribution, it is sometimes easier to find the potential by integration than the field, since the potential is a scalar. Having found the potential, one can always take the gradient to find the field.

14.2 Capacitance

A capacitor is a device that stores energy in an electric field. The simplest example consists of two parallel conducting plates. The energy is proportional to the square of the field strength, which is proportional to the charges on the plates. If we assume the plates carry charges that are the same in magnitude, $+q$ and $-q$, then the energy stored in the capacitor must be proportional to q^2. For historical reasons, we write the constant of proportionality as $1/2C$,

$$E = \frac{1}{2C}q^2.$$

The constant C is a geometrical property of the capacitor, called its capacitance. Based on this definition, the units of capacitance must be coulombs squared per joule, and this combination is more conveniently abbreviated as the farad, $1\ \mathrm{F} = 1\ \mathrm{C}^2/\mathrm{J}$.

Voltage is electrical potential energy per unit charge, so the voltage difference across a capacitor is related to the amount by which its energy would increase if we increased the absolute val-

ues of the charges on the plates from q to $q+\Delta q$:

$$\begin{aligned}V &= (E_{q+\Delta q} - E_q)/\Delta q \\ &= \frac{\Delta}{\Delta q}\left(\frac{1}{2C}q^2\right) \\ &= \frac{q}{C}\end{aligned}$$

Many books use this as the definition of capacitance. It follows from this relation that capacitances in parallel add, $C_{\text{equivalent}} = C_1 + C_2$, whereas when they are wired in series, it is their inverses that add, $C^{-1}_{\text{equivalent}} = C_1^{-1} + C_2^{-1}$.

14.3 Dielectrics

Many electrically insulating materials fall into a category known as dielectrics. Such materials can be modeled as containing many microscopic dipoles (molecules) that are randomly oriented but can become aligned when subjected to an external field. When we apply Gauss's law to a region of space in which a dielectric is present, the charge can have contributions both from free charges (such as the ones that flow in a circuit) measurable with measuring devices such as ammeters, but also from the bound, microscopic charges inside the dipoles. It can therefore be useful to rewrite Gauss's law as

$$\Phi_D = q_{\text{free}},$$

where

$$\mathbf{D} = \epsilon\mathbf{E}.$$

When the field is constant over time and not too strong, ϵ is approximately constant, and is a property of the material called its permittivity. In a vacuum, $\epsilon = 1/4\pi k$, referred to as ϵ_0, while a dielectric has $\epsilon > \epsilon_0$. With time-varying fields, most materials have permittivities that are highly frequency-dependent. For materials such as crystals, which have special directions defined by the regular atomic lattice, ϵ cannot be modeled as a scalar, and the relation between **D** and **E** becomes more complicated.

When a capacitor has the space between its electrodes filled with a dielectric, its capacitance is increased by the factor ϵ/ϵ_0.

At a boundary between two different materials, if there is no free charge at the boundary, the components of the fields $D_\perp$ and $E_\parallel$ are continuous.

14.4 Poisson's equation and Laplace's equation

Gauss's law, $\operatorname{div}\mathbf{E} = 4\pi k\rho$, can also be stated in terms of the potential. Since $\mathbf{E} = \nabla V$, we have $\operatorname{div}\nabla V = 4\pi k\rho$. If we work out the combination of operators $\operatorname{div}\nabla$ in a Cartesian coordinate system, we get $\partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$, which is called the Laplacian and notated ∇^2. The version of Gauss's law written in terms of the potential,

$$\nabla^2 V = 4\pi k\rho,$$

is called Poisson's equation, while in the special case of a vacuum, with $\rho = 0$, we have

$$\nabla^2 V = 0,$$

known as Laplace's equation. Many problems in electrostatics can be stated in terms of finding potential that satisfies Laplace's equation, usually with some set of *boundary conditions*. For example, if an infinite parallel-plate capacitor has plates parallel to the x-y plane at certain given potentials, then these plates form a boundary for the region between the plates, and Laplace's equation has a solution in this region of the form $V = az + b$. It's easy to verify that this is a solution of Laplace's equation, since all three of the partial derivatives vanish.

14.5 The method of images

A car's radio antenna is usually in the form of a whip sticking up above its metal roof. This is an example involving radio waves, which are time-varying electric and magnetic fields, but a

similar, simpler electrostatic example is the following. Suppose that we position a charge $q > 0$ at a distance ℓ from a conducting plane. What is the resulting electric field? The conductor has charges that are free to move, and due to the field of the charge q, we will end up with a net concentration of negative charge in the part of the plane near q. The field in the vacuum surrounding q will be a sum of fields due to q and fields due to these charges in the conducting plane. The problem can be stated as that of finding a solution to Poisson's equation with the boundary condition that $V = 0$ at the conducting plane. Figure 14.1/1 shows the kind of field lines we expect.

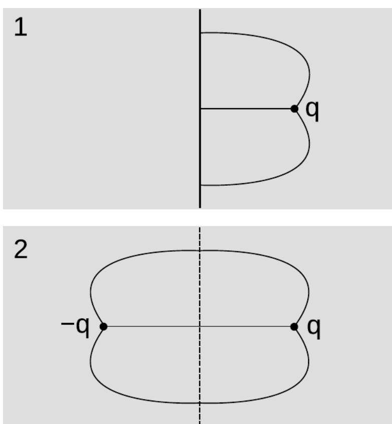


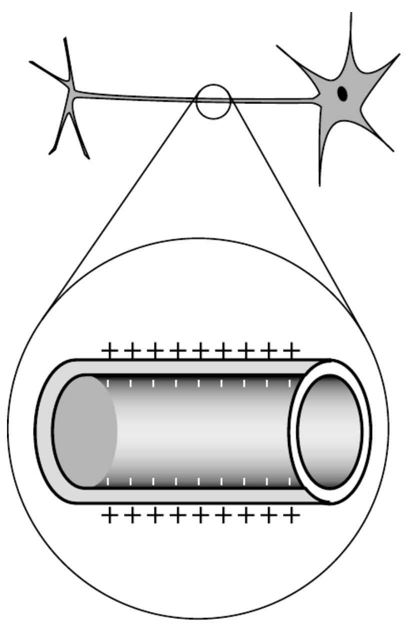
Figure 14.1: The method of images.

This looks like a very complicated problem, but there is trick that allows us to find a simple solution. We can convert the problem into an equivalent one in which the conductor isn't present, but a fictitious *image* charge $-q$ is placed at an equal distance behind the plane, like a reflection in a mirror, as in figure 14.1/2. The field is then simply the sum of the fields of the charges q and $-q$, so we can either add the field vectors or add the potentials. By symmetry, the field lines are perpendicular to the plane, so the plane is an surface of constant potential, as required.

Problems

14-a1 The gap between the electrodes in an automobile engine's spark plug is 0.060 cm. To produce an electric spark in a gasoline-air mixture, an electric field of 3.0×10^6 V/m must be achieved. On starting a car, what minimum voltage must be supplied by the ignition circuit? Assume the field is uniform. ✓
(b) The small size of the gap between the electrodes is inconvenient because it can get blocked easily, and special tools are needed to measure it. Why don't they design spark plugs with a wider gap?

14-a2 In our by-now-familiar neuron, the voltage difference between the inner and outer surfaces of the cell membrane is about $V_{out} - V_{in} = -70$ mV in the resting state, and the thickness of the membrane is about 6.0 nm (i.e., only about a hundred atoms thick). What is the electric field inside the membrane? ✓



Problem 14-a2.

14-a3 The neuron in the figure has been drawn fairly short, but some neurons in your spinal cord have tails (axons) up to a meter long. The inner and outer surfaces of the membrane act as the "plates" of a capacitor. (The fact that it has been rolled up into a cylinder has very little effect.) In order to function, the neuron must create a voltage difference V between the inner and outer surfaces of the membrane. Let the membrane's thickness, radius, and length be t, r, and L. (a) Calculate the energy that must be stored in the electric field for the neuron to do its job. (In real life, the membrane is made out of a substance called a dielectric, whose electrical properties increase the amount of energy that must be stored. For the sake of this analysis, ignore this fact.) ✓
(b) An organism's evolutionary fitness should be better if it needs less energy to operate its nervous system. Based on your answer to part a, what would you expect evolution to do to the dimensions t and r? What other constraints would keep these evolutionary trends from going too far?

14-a4 The figure shows a simplified diagram of an electron gun such as the one that creates the electron beam in a TV tube. Electrons that spontaneously emerge from the negative electrode (cathode) are then accelerated to the positive electrode, which has a hole in it. (Once they emerge through the hole, they will slow down. However, if the two electrodes are fairly close together, this slowing down is a small effect, because the attractive and repulsive forces experienced by the electron tend to cancel.)
(a) If the voltage difference between the electrodes is ΔV, what is the velocity of an electron as it emerges at B? Assume that its initial velocity, at A, is negligible, and that the velocity is nonrelativistic. (If you haven't read ch. 7 yet, don't worry about the remark about relativity.) ✓
(b) Evaluate your expression numerically for the case where ΔV=10 kV, and compare to the speed of light.

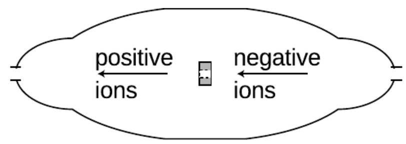
√ ▷ Solution, p. 204

14-a5 The figure shows a simplified diagram of a device called a tandem accelerator, used for accelerating beams of ions up to speeds on the order of 1-10% of the speed of light. (Since these velocities are not too big compared to c, you can use nonrelativistic physics throughout this problem.) The nuclei of these ions collide with the nuclei of atoms in a target, producing nuclear reactions for experiments studying the structure of nuclei. The outer shell of the accelerator is a conductor at zero voltage (i.e., the same voltage as the Earth). The electrode at the center, known as the "terminal," is at a high positive voltage, perhaps millions of volts. Negative ions with a charge of -1 unit (i.e., atoms with one extra electron) are produced offstage on the right, typically by chemical reactions with cesium, which is a chemical element that has a strong tendency to give away electrons. Relatively weak electric and magnetic forces are used to transport these -1 ions into the accelerator, where they are attracted to the terminal. Although the center of the terminal has a hole in it to let the ions pass through, there is a very thin carbon foil there that they must physically penetrate. Passing through the foil strips off some number of electrons, changing the atom into a positive ion, with a charge of $+n$ times the fundamental charge. Now that the atom is positive, it is repelled by the terminal, and accelerates some more on its way out of the accelerator.

(a) Find the velocity, v, of the emerging beam of positive ions, in terms of n, their mass m, the terminal voltage V, and fundamental constants. Neglect the small change in mass caused by the loss of electrons in the stripper foil. √

(b) To fuse protons with protons, a minimum beam velocity of about 11% of the speed of light is required. What terminal voltage would be needed in this case? √

(c) In the setup described in part b, we need a target containing atoms whose nuclei are single protons, i.e., a target made of hydrogen. Since hydrogen is a gas, and we want a foil for our target, we have to use a hydrogen compound, such as a plastic. Discuss what effect this would have on the experiment.



Problem 14-a5.

14-d1 A hydrogen atom is electrically neutral, so at large distances, we expect that it will create essentially zero electric field. This is not true, however, near the atom or inside it. Very close to the proton, for example, the field is very strong. To see this, think of the electron as a spherically symmetric cloud that surrounds the proton, getting thinner and thinner as we get farther away from the proton. (Quantum mechanics tells us that this is a more correct picture than trying to imagine the electron orbiting the proton.) Near the center of the atom, the electron cloud's field cancels out by symmetry, but the proton's field is strong, so the total field is very strong. The potential in and around the hydrogen atom can be approximated using an expression of the form $V = r^{-1}e^{-r}$. (The units come out wrong, because I've left out some constants.) Find the electric field corresponding to this potential, and comment on its behavior at very large and very small r.

▷ Solution, p. 204

14-d2 (a) Given that the on-axis field of a dipole at large distances is proportional to D/r^3, show that its potential varies as D/r^2. (Ignore positive and negative signs and numerical constants of proportionality.)

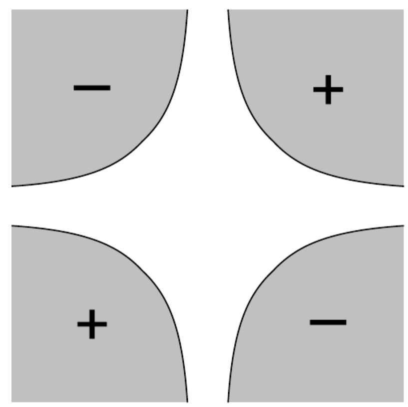
(b) Write down an exact expression for the potential of a two-charge dipole at an on-axis point, without assuming that the distance is large compared to the size of the dipole. Your expression will have to contain the actual charges and size of the dipole, not just its dipole moment. Now use

approximations to show that, at large distances, this is consistent with your answer to part a.

14-d3 A carbon dioxide molecule is structured like O-C-O, with all three atoms along a line. The oxygen atoms grab a little bit of extra negative charge, leaving the carbon positive. The molecule's symmetry, however, means that it has no overall dipole moment, unlike a V-shaped water molecule, for instance. Whereas the potential of a dipole of magnitude D is proportional to D/r^2, (see problem 14-d2), it turns out that the potential of a carbon dioxide molecule at a distant point along the molecule's axis equals b/r^3, where r is the distance from the molecule and b is a constant (cf. problem 13-g3). What would be the electric field of a carbon dioxide molecule at a point on the molecule's axis, at a distance r from the molecule?
√

14-d4 A proton is in a region in which the electric field is given by $E = a + bx^3$. If the proton starts at rest at $x_1 = 0$, find its speed, v, when it reaches position x_2. Give your answer in terms of a, b, x_2, and e and m, the charge and mass of the proton.
√

14-g1 The figure shows a vacuum chamber surrounded by four metal electrodes shaped like hyperbolas. (Yes, physicists do sometimes ask their university machine shops for things machined in mathematical shapes like this. They have to be made on computer-controlled mills.) We assume that the electrodes extend far into and out of the page along the unseen z axis, so that by symmetry, the electric fields are the same for all z. The problem is therefore effectively two-dimensional. Two of the electrodes are at voltage $+V_o$, and the other two at $-V_o$, as shown. The equations of the hyperbolic surfaces are $|xy| = b^2$, where b is a constant. (We can interpret b as giving the locations $x = \pm b$, $y = \pm b$ of the four points on the surfaces that are closest to the central axis.) There is no obvious, pedestrian way to determine the field or potential in the central vacuum region, but there's a trick that works: with a little mathematical insight, we see that the potential $V = V_o b^{-2} xy$ is consistent with all the given information. (Mathematicians could prove that this solution was unique, but a physicist knows it on physical grounds: if there were two different solutions, there would be no physical way for the system to decide which one to do!)
(a) Find the field in the vacuum region.
(b) Sketch the field as a "sea of arrows."
√



Problem 14-g1.

14-g2 (a) A certain region of three-dimensional space has a potential that varies as $V = br^2$, where r is the distance from the origin. Find the field. √
(b) Write down another potential that gives exactly the same field.

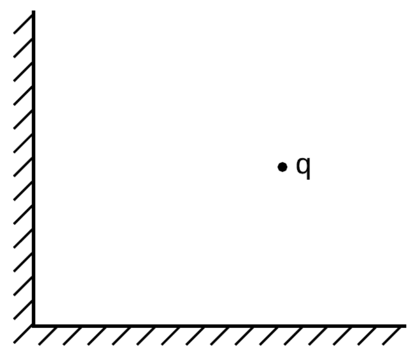
14-j1 Find the capacitance of the surface of the earth, assuming there is an outer spherical "plate" at infinity. (In reality, this outer plate would just represent some distant part of the universe to which we carried away some of the earth's charge in order to charge up the earth.)
√

14-j2 Find the energy stored in a capacitor in terms of its capacitance and the voltage difference across it.
√

14-j3 A capacitor has parallel plates of area A, separated by a distance h. If there is a vacuum between the plates, then Gauss's law gives $E = 4\pi k\sigma = 4\pi kq/A$ for the field between the plates, and combining this with $E = V/h$, we find $C = q/V = (1/4\pi k)A/h$. (a) Generalize this derivation to the case where there is a dielectric between the plates. (b) Suppose we have a list of possible materials we could choose as dielectrics, and we wish to construct a capacitor that will have the highest possible energy density, U_e/v, where v is the volume. For each dielectric, we know its permittivity ϵ, and also the maximum electric field E it can sustain without breaking down and allowing sparks to cross between the plates. Write the maximum energy density in terms of these two variables, and determine a figure of merit that could be used to decide which material would be the best choice.

14-m1 A charged particle of mass m and charge q is below a horizontal conducting plane. We wish to find the distance ℓ between the particle and the plane so that the particle will be in equilibrium, with its weight supported by electrostatic forces.
(a) Determine as much as possible about the form of the answer based on units.
(b) Find the full result for ℓ.
(c) Show that the equilibrium is unstable.

14-m2 A point charge q is situated in the empty space inside a corner formed by two perpendicular half-planes made of sheets of metal. Let the sheets lie in the y-z and x-z planes, so that the charge's distances from the planes are x and y. Both x and y are positive. The charge will accelerate due to the electrostatic forces exerted by the sheets. We wish to find the direction θ in which it will accelerate, expressed as an angle counterclockwise from the negative x axis, so that $0 < \theta < \pi/2$.
(a) Determine as much as possible about the form of the answer based on units.
(b) Find the full result for θ.



Problem 14-m2.

15 Circuits

This is not a textbook. It's a book of problems meant to be used along with a textbook. Although each chapter of this book starts with a brief summary of the relevant physics, that summary is not meant to be enough to allow the reader to actually learn the subject from scratch. The purpose of the summary is to show what material is needed in order to do the problems, and to show what terminology and notation are being used.

15.1 Current

Electric current is defined as the rate of flow of charge through a boundary, $I = \mathrm{d}q/\,\mathrm{d}t$. Its units of coulombs/second are more conveniently abbreviated as amperes, 1 A=1 C/s. Current is measured by an ammeter, which allows current to flow through itself. An ammeter is wired in series with a circuit, which requires breaking the circuit in order to insert the meter. An ideal ammeter acts like a perfect conductor.

The power dissipated, transformed, or released in an electric circuit element is given by $P = I\Delta V$.

15.2 Resistance

Current will not flow at all through a perfect insulator. When a material is neither a perfect insulator nor a perfect insulator, then current can flow through it, and the result in terms of energy is that electrical energy is transformed into heat. For many materials, under some fairly large range of electric fields, the density of current is proportional to the electric field. When a two-terminal device is formed from such a material, and a voltage difference is applied across it, then the current flowing through it is given by Ohm's law, $I = \Delta V/R$, where R, called the resistance, depends on both the geometry of the device and the material of which it is constructed. Despite the name, Ohm's law is not a law of nature, and it is often violated. Some substances, such as gases, never obey Ohm's law; we say that they are not "ohmic." The units of resistance are abbreviated as ohms, $1\ \Omega = 1\ \mathrm{V/A}$.

Resistances in series add, $R_{\text{equivalent}} = R_1 + R_2$, while in parallel $R^{-1}_{\text{equivalent}} = R_1^{-1} + R_2^{-1}$.

15.3 The loop and junction rules

Kirchoff's junction rule is a statement of conservation of charge. It says that the sum of the currents flowing into any junction in a circuit must be zero (if the junction has no way to store charge).

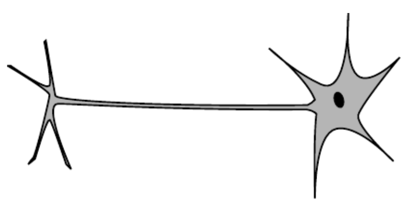
Kirchoff's loop rule is a statement of conservation of energy. For any loop in a circuit, the sum of the voltage drops must be zero.

Problems

15-a1 In a wire carrying a current of 1.0 pA, how long do you have to wait, on the average, for the next electron to pass a given point? Express your answer in units of microseconds.

▷ Solution, p. 204

15-a2 Referring back to our old friend the neuron from problem 13-a1 on page 154, let's now consider what happens when the nerve is stimulated to transmit information. When the blob at the top (the cell body) is stimulated, it causes Na^+ ions to rush into the top of the tail (axon). This electrical pulse will then travel down the axon, like a flame burning down from the end of a fuse, with the Na^+ ions at each point first going out and then coming back in. If 10^{10} Na^+ ions cross the cell membrane in 0.5 ms, what amount of current is created? ✓



Problem 15-a2.

15-a3 Lightning discharges a cloud during an electrical storm. Suppose that the current in the lightning bolt varies with time as $I = bt$, where b is a constant. Find the cloud's charge as a function of time. ✓

15-a4 (a) You take an LP record out of its sleeve, and it acquires a static charge of 1 nC. You play it at the normal speed of $33\frac{1}{3}$ r.p.m., and the charge moving in a circle creates an electric current. What is the current, in amperes? ✓

(b) Although the planetary model of the atom can be made to work with any value for the radius of the electrons' orbits, more advanced models that we will study later in this course predict definite radii. If the electron is imagined as circling around the proton at a speed of 2.2×10^6 m/s, in an orbit with a radius of 0.05 nm, what electric current is created? ✓

Problem 15-a4.

15-a5 A silk thread is uniformly charged by rubbing it with llama fur. The thread is then dangled vertically above a metal plate and released. As each part of the thread makes contact with the conducting plate, its charge is deposited onto the plate. Since the thread is accelerating due to gravity, the rate of charge deposition increases with time, and by time t the cumulative amount of charge is $q = ct^2$, where c is a constant. (a) Find the current flowing onto the plate. ✓
(b) Suppose that the charge is immediately carried away through a resistance R. Find the power dissipated as heat. ✓

15-a6 In AM (amplitude-modulated) radio, an audio signal $f(t)$ is multiplied by a sine wave $\sin \omega t$ in the megahertz frequency range. For simplicity, let's imagine that the transmitting antenna is a whip, and that charge goes back and forth between the top and bottom. Suppose that, during a certain time interval, the audio signal varies linearly with time, giving a charge $q = (a + bt) \sin \omega t$ at the top of the whip and $-q$ at the bottom. Find the current as a function of time. ✓

15-d1 If a typical light bulb draws about 900 mA from a 110-V household circuit, what is its

resistance? (Don't worry about the fact that it's alternating current.)

$\checkmark$

15-d2 (a) Express the power dissipated by a resistor in terms of R and ΔV only, eliminating I. $\checkmark$

(b) Electrical receptacles in your home are mostly 110 V, but circuits for electric stoves, air conditioners, and washers and driers are usually 220 V. The two types of circuits have differently shaped receptacles. Suppose you rewire the plug of a drier so that it can be plugged in to a 110 V receptacle. The resistor that forms the heating element of the drier would normally draw 200 W. How much power does it actually draw now?

$\checkmark$

15-d3 A resistor has a voltage difference ΔV across it, causing a current I to flow.

(a) Find an equation for the power it dissipates as heat in terms of the variables I and R only, eliminating ΔV. $\checkmark$

(b) If an electrical line coming to your house is to carry a given amount of current, interpret your equation from part a to explain whether the wire's resistance should be small, or large.

15-d4 We have referred to resistors *dissipating* heat, i.e., we have assumed that $P = I\Delta V$ is always greater than zero. Could $I\Delta V$ come out to be negative for a resistor? If so, could one make a refrigerator by hooking up a resistor in such a way that it absorbed heat instead of dissipating it?

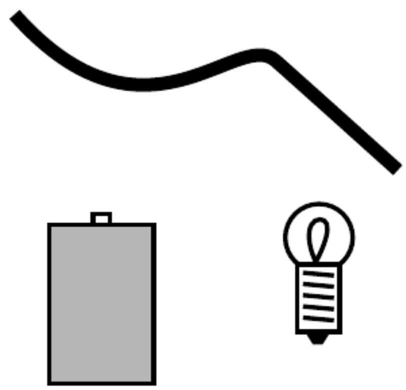
15-d5 What resistance values can be created by combining a 1 kΩ resistor and a 10 kΩ resistor?

▷ Solution, p. 204

15-d6 Wire is sold in a series of standard diameters, called "gauges." The difference in diameter between one gauge and the next in the series is about 20%. How would the resistance of a given length of wire compare with the resistance of the same length of wire in the next gauge in the series?

$\checkmark$

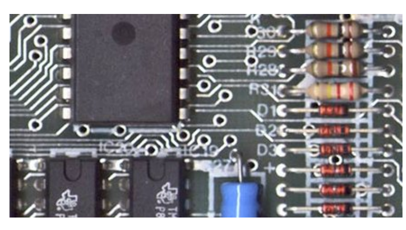
15-g1 You are given a battery, a flashlight bulb, and a single piece of wire. Draw at least two configurations of these items that would result in lighting up the bulb, and at least two that would not light it. (Don't draw schematics.) If you're not sure what's going on, borrow the materials from your instructor and try it. Note that the bulb has two electrical contacts: one is the threaded metal jacket, and the other is the tip (at the bottom in the figure). [Problem by Arnold Arons.]



Problem 15-g1.

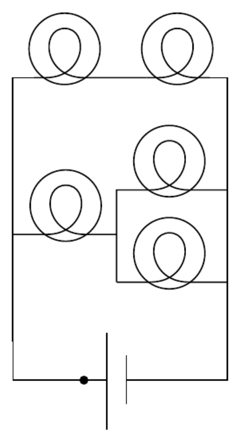
15-g2 You have to do different things with a circuit to measure current than to measure a voltage difference. Which would be more practical for a printed circuit board, in which the wires are actually strips of metal embedded inside the board?

▷ Solution, p. 204



Problem 15-g2.

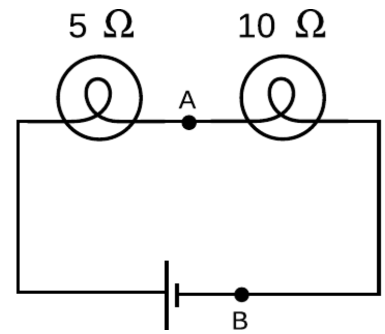
15-g3 The figure shows a circuit containing five lightbulbs connected to a battery. Suppose you're going to connect one probe of a voltmeter to the circuit at the point marked with a dot. How many unique, nonzero voltage differences could you measure by connecting the other probe to other wires in the circuit?



Problems 15-g3 and 15-g4.

15-g4 The lightbulbs in the figure are all identical. If you were inserting an ammeter at various places in the circuit, how many unique currents could you measure? If you know that the current measurement will give the same number in more than one place, only count that as one unique current.

15-g5 In the figure, the battery is 9 V.
(a) What are the voltage differences across each light bulb? ✓
(b) What current flows through each of the three components of the circuit? ✓
(c) If a new wire is added to connect points A and B, how will the appearances of the bulbs change? What will be the new voltages and currents?
(d) Suppose no wire is connected from A to B, but the two bulbs are switched. How will the results compare with the results from the original setup as drawn?

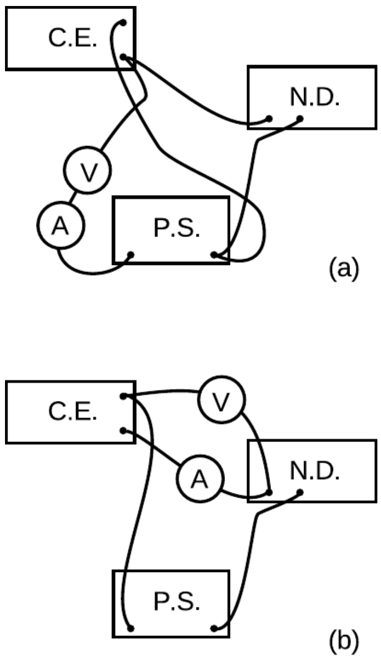


Problem 15-g5.

15-g6 A student in a biology lab is given the following instructions: "Connect the cerebral eraser (C.E.) and the neural depolarizer (N.D.) in parallel with the power supply (P.S.). (Under no circumstances should you ever allow the cerebral eraser to come within 20 cm of your head.) Connect a voltmeter to measure the voltage across the cerebral eraser, and also insert an ammeter in the circuit so that you can make sure you don't put more than 100 mA through the neural depolarizer." The diagrams show two lab groups' attempts to follow the instructions. (a) Translate diagram a into a standard-style schematic. What is correct and incorrect about this group's setup? (b) Do the same for diagram b.

15-g7 A 1.0 Ω toaster and a 2.0 Ω lamp are connected in parallel with the 110-V supply of your house. (Ignore the fact that the voltage is AC rather than DC.)
(a) Draw a schematic of the circuit.
(b) For each of the three components in the circuit, find the current passing through it and the voltage drop across it. ✓
(c) Suppose they were instead hooked up in series. Draw a schematic and calculate the same things. ✓

15-g8 The heating element of an electric stove is connected in series with a switch that opens and closes many times per second. When you



Problem 15-g6.

turn the knob up for more power, the fraction of the time that the switch is closed increases. Suppose someone suggests a simpler alternative for controlling the power by putting the heating element in series with a variable resistor controlled by the knob. (With the knob turned all the way clockwise, the variable resistor's resistance is nearly zero, and when it's all the way counterclockwise, its resistance is essentially infinite.) (a) Draw schematics. (b) Why would the simpler design be undesirable?

15-g9 You have a circuit consisting of two unknown resistors in series, and a second circuit consisting of two unknown resistors in parallel.
(a) What, if anything, would you learn about the resistors in the series circuit by finding that the currents through them were equal?
(b) What if you found out the voltage differences across the resistors in the series circuit were equal?
(c) What would you learn about the resistors in the parallel circuit from knowing that the currents were equal?
(d) What if the voltages in the parallel circuit were equal?

15-g10 How many different resistance values can be created by combining three unequal resistors? (Don't count possibilities in which not all the resistors are used, i.e., ones in which there is zero current in one or more of them.)

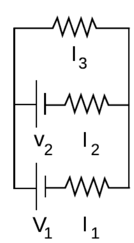
15-g11 Suppose six identical resistors, each with resistance R, are connected so that they form the edges of a tetrahedron (a pyramid with three sides in addition to the base, i.e., one less side than an Egyptian pyramid). What resistance value or values can be obtained by making connections onto any two points on this arrangement?

▷ Solution, p. 204

15-g12 A person in a rural area who has no electricity runs an extremely long extension cord to a friend's house down the road so she can run an electric light. The cord is so long that its resistance, x, is not negligible. Show that the lamp's brightness is greatest if its resistance, y, is equal to x. Explain physically why the lamp is dim for values of y that are too small or too large.

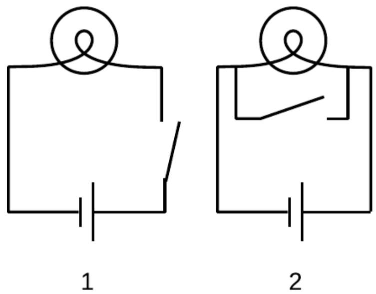
15-g13 All three resistors have the same resistance, R. Find the three unknown currents in terms of V_1, V_2, and R.

√



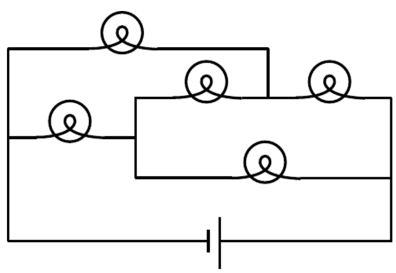
Problem 15-g13.

15-g14 The figure shows two possible ways of wiring a flashlight with a switch. Both will serve to turn the bulb on and off, although the switch functions in the opposite sense. Why is method 1 preferable?



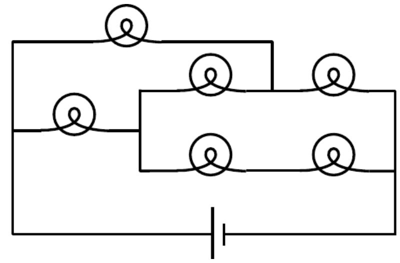
Problem 15-g14.

15-g15 The bulbs are all identical. Which one doesn't light up?



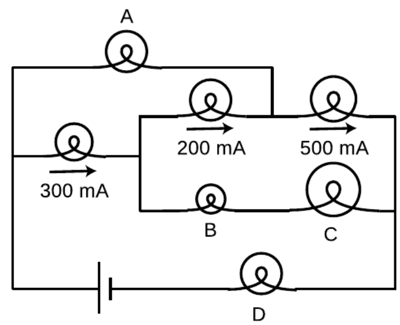
Problem 15-g15.

15-g16 Each bulb has a resistance of one ohm. How much power is drawn from the one-volt battery? ✓



Problem 15-g16.

15-g17 The bulbs all have unequal resistances. Given the three currents shown in the figure, find the currents through bulbs A, B, C, and D.



Problem 15-g17.

15-g18 It's fairly common in electrical circuits for additional, undesirable resistances to occur because of factors such as dirty, corroded, or loose connections. Suppose that a device with resistance R normally dissipates power P, but due to an additional series resistance r the *total* power is reduced to P'. We might, for example, detect this change because the battery powering our device ran down more quickly than normal.
(a) Find the unknown resistance r. ✓
(b) Check that the units of your result make sense.
(c) Check that your result makes sense in the special cases $P' = P$ and $P' = 0$.
(d) Suppose we redefine P' as the useful power dissipated in R. For example, this would be the change we would notice because a flashlight was dimmer. Find r. ✓

15-j1 Suppose a parallel-plate capacitor is built so that a slab of dielectric material can be slid in or out. (This is similar to the way a stud finder works.) We insert the dielectric, hook the capacitor up to a battery to charge it, and then use an ammeter and a voltmeter to observe

what happens when the dielectric is withdrawn. Predict the changes observed on the meters, and correlate them with the expected change in capacitance. Discuss the energy transformations involved, and determine whether positive or negative work is done in removing the dielectric.

Problem 15-j1.

15-j2 Repeat problem 15-j1, but with one change in the procedure: after we charge the capacitor, we open the circuit, and then continue with the observations.

16 Basics of relativity

This is not a textbook. It's a book of problems meant to be used along with a textbook. Although each chapter of this book starts with a brief summary of the relevant physics, that summary is not meant to be enough to allow the reader to actually learn the subject from scratch. The purpose of the summary is to show what material is needed in order to do the problems, and to show what terminology and notation are being used.

16.1 The Lorentz transformation

There is a saying among biologists that without evolution, nothing in biology makes sense. Similarly, it is impossible to make sense out of electricity and magnetism, beyond simple electrostatics and DC circuits, without understanding a few basic ideas about Einstein's theory of special relativity.

According to Galileo and Newton, motion is relative but time is absolute. This theory of time and space is called Galilean relativity. According to Galilean relativity, observers in different states of motion will have position and time coordinates that relate to one another in the manner shown in figure 16.1.

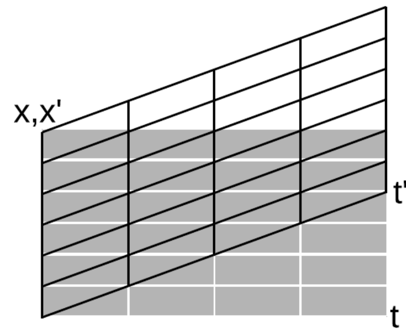


Figure 16.1: The relationship between time and space coordinates in two different frames of reference, according to Galilean relativity.

Experiments show that this absoluteness of time is only an approximation, valid at low speeds. At high speeds, or with sufficiently precise experiments, we find that time is relative. Although this idea dates back to a 1905 paper by Einstein, and certain types of indirect experimental evidence go back as far as the 19th century, modern technology has made it easier to demonstrate this in more direct and conceptually simple experiments. In 2010, for example, Chou *et al.* succeeded in building an atomic clock accurate enough to detect an effect at speeds as low as 10 m/s. The correct relationship between time and space in different frames of reference, proposed mathematically by Lorentz and interpreted correctly by Einstein, is called the Lorentz transformation, figure 16.2.

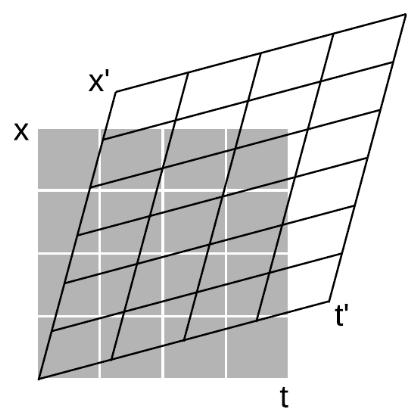


Figure 16.2: The Lorentz transformation.

The Lorentz transformation shown in the figure has a simple symmetry with respect to a flip across the diagonal. This symmetry is present only when we use units specially adapted to relativity. In such units, the slope of the 45-degree diagonal is a special speed having the value 1, and time and space are measured in the same units. In SI units, this special speed is notated

c, and it has a defined value, equal to approximately 3.0×10^8 m/s. One of the predictions of relativity is that anything without mass must move at this speed.

Since light was historically the first example encountered, c is often referred to as the speed of light, but relativity tells us that it is better to think of c as a kind of conversion factor between space and time.

The speed c has the following fundamentally important properties. It is the only speed that observers in different states of motion agree on. It is the speed at which massless objects always travel, and it is an ultimate speed limit for massive objects. It is the maximum speed of propagations for signals or for any mechanism of cause and effect.

The Lorentz transformation can be expressed algebraically, although it would be a distraction to do so here. Its form is determined entirely by the facts that (1) the slope of the t' axis is the velocity of one observer relative to the other, (2) the main diagonal keeps the same slope, and (3) the area of the boxes is preserved in the transformation.

16.2 Length contraction and time dilation

From these facts about the Lorentz transformation, it can be shown that different observers disagree about lengths and times in the following way. Let $\gamma = 1/\sqrt{1-v^2}$, where v is the velocity of one observer relative to another. (In SI units, substitute v/c for v.)

A clock appears to run fastest to an observer at rest relative to the clock. According to other observers, the clock's rate is lower by a factor of γ.

A meter stick appears longest to an observer at rest relative to the stick. An observer in motion parallel to the stick measures the stick to have been shortened by a factor of γ.

Problems

16-a1 The figure illustrates a Lorentz transformation using the conventions described in sec. 16.1, p. 177. For simplicity, the transformation chosen is one that lengthens one diagonal by a factor of 2. Since Lorentz transformations preserve area, the other diagonal is shortened by a factor of 2. Let the original frame of reference, depicted with the square, be A, and the new one B. (a) By measuring with a ruler on the figure, show that the velocity of frame B relative to frame A is $0.6c$. (b) Print out a copy of the page. With a ruler, draw a third parallelogram that represents a second successive Lorentz transformation, one that lengthens the long diagonal by another factor of 2. Call this third frame C. Use measurements with a ruler to determine frame C's velocity relative to frame A. Does it equal double the velocity found in part a? Explain why it should be expected to turn out the way it does.

$\checkmark$

16-a2 Astronauts in three different spaceships are communicating with each other. Those aboard ships A and B agree on the rate at which time is passing, but they disagree with the ones on ship C.
(a) Alice is aboard ship A. How does she describe the motion of her own ship, in its frame of reference?
(b) Describe the motion of the other two ships according to Alice.
(c) Give the description according to Betty, whose frame of reference is ship B.
(d) Do the same for Cathy, aboard ship C.

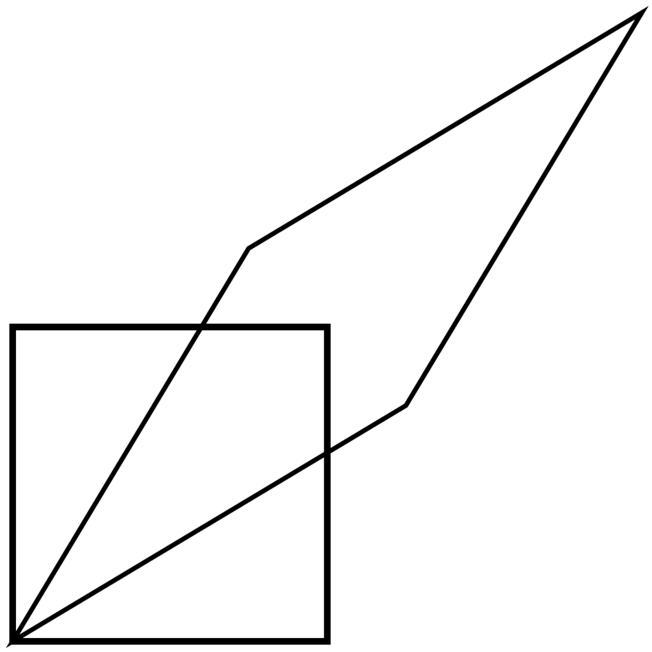
16-a3 What happens in the equation for γ when you put in a negative number for v? Explain what this means physically, and why it makes sense.

16-a4 The Voyager 1 space probe, launched in 1977, is moving faster relative to the earth than any other human-made object, at 17,000 meters per second.
(a) Calculate the probe's γ.
(b) Over the course of one year on earth, slightly less than one year passes on the probe. How much less? (There are 31 million seconds in a year.)

$\checkmark$

16-a5 The earth is orbiting the sun, and therefore is contracted relativistically in the direction of its motion. Compute the amount by which its diameter shrinks in this direction. $\checkmark$

16-a6 (a) Show that for $v = (3/5)c$, γ comes out to be a simple fraction.
(b) Find another value of v for which γ is a simple fraction.



Problem 16-a1.

17 Electromagnetism

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17.1 Electromagnetism

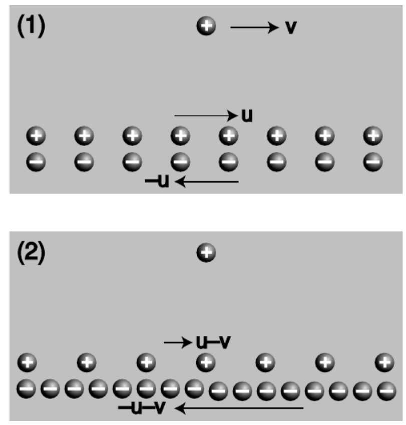


Figure 17.1: A charged particle and a current, seen in two different frames of reference. The second frame is moving at velocity v with respect to the first frame, so all the velocities have v subtracted from them (approximately).

The top panel of figure 17.1 shows a charged particle moving to the right, parallel to a current formed by two countermoving lines of opposite charge, moving at velocities $\pm u$. The two lines of charge are drawn offset from each other to make them easy to distinguish, but we think of them as coinciding, so that the line is electrically neutral over all, much like a current-carrying copper wire. Based on our knowledge of electrostatics, we would expect the lone charge to feel zero force, since the neutral "wire" has no electric field.

The bottom panel of the figure shows the same situation in the rest frame of the lone charge. Although velocities do not exactly add and subtract in special relativity as they would in Galilean relativity (problem 16-a1, p. 179), they approximately do if the velocities are not too big, so that the velocities of the two lines of charge are approximately $u - v$ and $-u - v$. Since the magnitudes of these velocities are unequal, the length contractions are unequal, and the "wire" is charged, according to an observer in this frame. Therefore the lone charge feels an attractive (downward) electrical force.

The descriptions in the two frames of reference is inconsistent, so we introduce a force in the original frame. A moving charge always interacts with other moving charges through such a force, called a magnetic force. Thus if we had only known about electrical interactions, relativity would have compelled us to introduce magnetic interactions as well. Relativity *unifies* the electrical and magnetic interactions as two sides of the same coin. The unified theory of electricity and magnetism is called electromagnetism.

17.2 The magnetic field

The magnetic force acting on a charged particle is $q\mathbf{v} \times \mathbf{B}$, where $\mathbf{B}$ is the magnetic field. This is partly a definition of $\mathbf{B}$ and partly a prediction about how the force depends on $\mathbf{v}$. The units of the electric field are N·s/C·m, which can be abbreviated as tesla, 1 T = 1 N·s/C·m.

Empirically, we find that the magnetic field has no sources or sinks. Gauss' law for magnetism is

$$\Phi_B = 0.$$

In other words, there are no magnetic monopoles. There are, however, magnetic

dipoles. Subatomic particles such as electrons and neutrons have magnetic dipole moments, as do some molecules. As a standard of comparison, the magnetic dipole moment $\mathbf{m}$ of a loop of current has magnitude $m = IA$, and is in the (right-handed) direction perpendicular to the loop. The energy of a magnetic dipole in an external magnetic field is $-\mathbf{m} \cdot \mathbf{B}$, and the torque acting on it is $\mathbf{m} \times \mathbf{B}$.

The energy of the magnetic field is

$$\mathrm{d}U_m = \frac{c^2}{8\pi k} B^2 \,\mathrm{d}v.$$

When a static magnetic field is caused by a current loop, the *Biot-Savart law*,

$$\mathrm{d}\mathbf{B} = \frac{kI \,\mathrm{d}\boldsymbol{\ell} \times \mathbf{r}}{c^2 r^3},$$

gives the field when we integrate over the loop.

Ampère's law is another way of relating static magnetic fields to the static currents that created them, and it is more easily extended to nonstatic fields than is the Biot-Savart law. Ampère's law states that the *circulation* of the magnetic field,

$$\Gamma_B = \int \mathbf{B} \cdot \mathrm{d}\mathbf{s},$$

around the edge of a surface is related to the current $I_{through}$ passing through the surface,

$$\Gamma = \frac{4\pi k}{c^2} \, I_{through}.$$

Problems

17-a1 A particle with a charge of 1.0 C and a mass of 1.0 kg is observed moving past point P with a velocity $(1.0\text{ m/s})\hat{\mathbf{x}}$. The electric field at point P is $(1.0\text{ V/m})\hat{\mathbf{y}}$, and the magnetic field is $(2.0\text{ T})\hat{\mathbf{y}}$. Find the force experienced by the particle.
$\checkmark$

17-a2 For a positively charged particle moving through a magnetic field, the directions of the **v**, **B**, and **F** vectors are related by a right-hand rule:

v along the fingers, with the hand flat
B along the fingers, with the knuckles bent
F along the thumb

Make a three-dimensional model of the three vectors using pencils or rolled-up pieces of paper to represent the vectors assembled with their tails together. Make all three vectors perpendicular to each other. Now write down every possible way in which the rule could be rewritten by scrambling up the three symbols **v**, **B**, and **F**. Referring to your model, which are correct and which are incorrect?

17-a3 A charged particle is released from rest. We see it start to move, and as it gets going, we notice that its path starts to curve. Can we tell whether this region of space has $\mathbf{E} \neq 0$, or $\mathbf{B} \neq 0$, or both? Assume that no other forces are present besides the possible electrical and magnetic ones, and that the fields, if they are present, are uniform.

17-a4 A charged particle is in a region of space in which there is a uniform magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$. There is no electric field, and no other forces act on the particle. In each case, describe the future motion of the particle, given its initial velocity.
(a) $\mathbf{v}_\text{o} = 0$
(b) $\mathbf{v}_\text{o} = (1\text{ m/s})\hat{\mathbf{z}}$
(c) $\mathbf{v}_\text{o} = (1\text{ m/s})\hat{\mathbf{y}}$

17-a5 (a) A line charge, with charge per unit length λ, moves at velocity v along its own length. How much charge passes a given point in time dt? What is the resulting current?
(b) Show that the units of your answer in part a work out correctly.
Remark: This constitutes a physical model of an electric current, and it would be a physically realistic model of a beam of particles moving in a vacuum, such as the electron beam in a television tube. It is not a physically realistic model of the motion of the electrons in a current-carrying wire, or of the ions in your nervous system; the motion of the charge carriers in these systems is much more complicated and chaotic, and there are charges of both signs, so that the total charge is zero. But even when the model is physically unrealistic, it still gives the right answers when you use it to compute magnetic effects. This is a remarkable fact, which we will not prove. The interested reader is referred to E.M. Purcell, *Electricity and Magnetism*, McGraw Hill, 1963.

17-a6 Two parallel wires of length L carry currents I_1 and I_2. They are separated by a distance R, and we assume R is much less than L, so that our results for long, straight wires are accurate. The goal of this problem is to compute the magnetic forces acting between the wires.
(a) Neither wire can make a force on *itself*. Therefore, our first step in computing wire 1's force on wire 2 is to find the magnetic field made only by wire 1, in the space *occupied* by wire 2. Express this field in terms of the given quantities. $\checkmark$
(b) Let's model the current in wire 2 by pretending that there is a line charge inside it, possessing density per unit length λ_2 and moving at velocity v_2. Relate λ_2 and v_2 to the current I_2, using the result of problem 17-a5a. Now find the magnetic force wire 1 makes on wire 2, in terms of I_1, I_2, L, and R.
(c) Show that the units of the answer to part b work out to be newtons.

17-a7 Suppose a charged particle is moving through a region of space in which there is an electric field perpendicular to its velocity vector, and also a magnetic field perpendicular to

both the particle's velocity vector and the electric field. Show that there will be one particular velocity at which the particle can be moving that results in a total force of zero on it. Relate this velocity to the magnitudes of the electric and magnetic fields. (Such an arrangement, called a velocity filter, is one way of determining the speed of an unknown particle.)

17-a8 The following data give the results of two experiments in which charged particles were released from the same point in space, and the forces on them were measured:

$$q_1 = 1\ \mu\text{C}\,, \qquad q_2 = -2\ \mu\text{C}\,,$$
$$\mathbf{v}_1 = (1\text{ m/s})\hat{\mathbf{x}}\,, \qquad \mathbf{v}_2 = (-1\text{ m/s})\hat{\mathbf{x}}\,,$$
$$\mathbf{F}_1 = (-1\text{ mN})\hat{\mathbf{y}} \qquad \mathbf{F}_2 = (-2\text{ mN})\hat{\mathbf{y}}$$

The data are insufficient to determine the magnetic field vector; demonstrate this by giving two different magnetic field vectors, both of which are consistent with the data.

17-a9 The following data give the results of two experiments in which charged particles were released from the same point in space, and the forces on them were measured:

$$q_1 = 1\text{ nC}\,, \qquad q_2 = 1\text{ nC}\,,$$
$$\mathbf{v}_1 = (1\text{ m/s})\hat{\mathbf{z}}\,, \qquad \mathbf{v}_2 = (3\text{ m/s})\hat{\mathbf{z}}\,,$$
$$\mathbf{F}_1 = (5\text{ pN})\hat{\mathbf{x}} + (2\text{ pN})\hat{\mathbf{y}} \qquad \mathbf{F}_2 = (10\text{ pN})\hat{\mathbf{x}} + (4\text{ pN})\hat{\mathbf{y}}$$

Is there a nonzero electric field at this point? A nonzero magnetic field?

17-a10 This problem is a continuation of problem 17-a6. Note that the answer to problem 17-a6b is given on page **??**.

(a) Interchanging the 1's and 2's in the answer to problem 17-a6b, what is the magnitude of the magnetic force from wire 2 acting on wire 1? Is this consistent with Newton's third law?

(b) Suppose the currents are in the same direction. Make a sketch, and use the right-hand rule to determine whether wire 1 pulls wire 2 towards it, or pushes it away.

(c) Apply the right-hand rule again to find the direction of wire 2's force on wire 1. Does this agree with Newton's third law?

(d) What would happen if wire 1's current was in the opposite direction compared to wire 2's?

17-a11 (a) In the photo, magnetic forces cause a beam of electrons to move in a circle. The beam is created in a vacuum tube, in which a small amount of hydrogen gas has been left. A few of the electrons strike hydrogen molecules, creating light and letting us see the beam. A magnetic field is produced by passing a current (meter) through the circular coils of wire in front of and behind the tube. In the bottom figure, with the magnetic field turned on, the force perpendicular to the electrons' direction of motion causes them to move in a circle. infer the direction of the magnetic field from the motion of the electron beam. (The answer is given in the answer to the self-check on that page.)

(b) Based on your answer to part a, find the direction of the currents in the coils.

(c) What direction are the electrons in the coils going?

(d) Are the currents in the coils repelling the currents consisting of the beam inside the tube, or attracting them? Check your answer by comparing with the result of problem 17-a10.

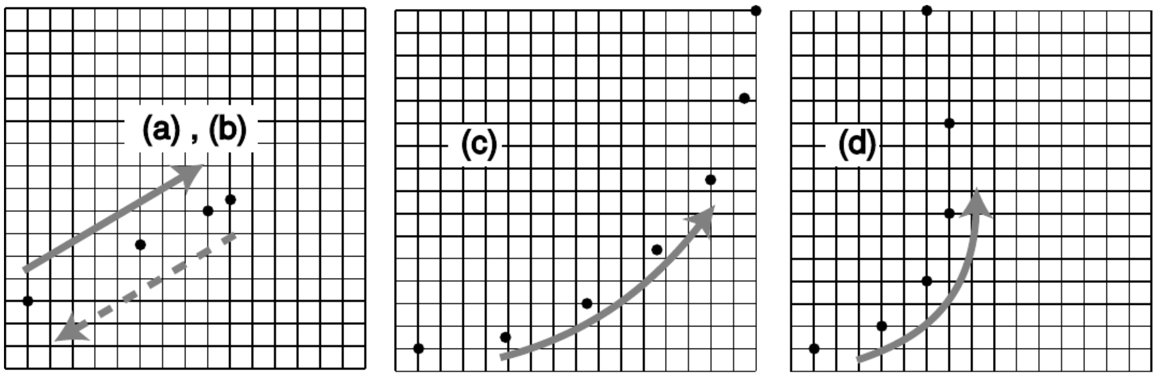
17-a12 A charged particle of mass m and charge q moves in a circle due to a uniform magnetic field of magnitude B, which points perpendicular to the plane of the circle.

(a) Assume the particle is positively charged. Make a sketch showing the direction of motion and the direction of the field, and show that the resulting force is in the right direction to produce circular motion.

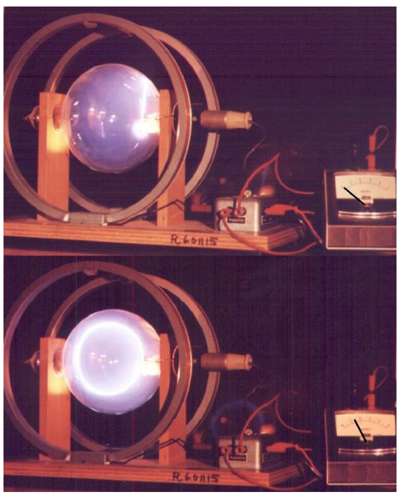
(b) Find the radius, r, of the circle, in terms of m, q, v, and B. √

(c) Show that your result from part b has the right units.

(d) Discuss all four variables occurring on the right-hand side of your answer from part b. Do they make sense? For instance, what should happen to the radius when the magnetic field is made stronger? Does your equation behave this way?



Problem 17-a13.


Problem 17-a11.

(e) Restate your result so that it gives the particle's angular frequency, ω, in terms of the other variables, and show that v drops out. ✓

Remark: A charged particle can be accelerated in a circular device called a cyclotron, in which a magnetic field is what keeps them from going off straight. This frequency is therefore known as the cyclotron frequency. The particles are accelerated by other forces (electric forces), which are AC. As long as the electric field is operated at the correct cyclotron frequency for the type of particles being manipulated, it will stay in sync with the particles, giving them a shove in the right direction each time they pass by. The particles are speeding up, so this only works because the cyclotron frequency is independent of velocity.

17-a13 Each figure represents the motion of a positively charged particle. The dots give the particles' positions at equal time intervals. In each case, determine whether the motion was caused by an electric force, a magnetic force, or a frictional force, and explain your reasoning. If possible, determine the direction of the magnetic or electric field. All fields are uniform. In (a), the particle stops for an instant at the upper right, but then comes back down and to the left, retracing the same dots. In (b), it stops on the upper right and stays there.

17-a14 One model of the hydrogen atom has the electron circling around the proton at a speed of 2.2×10^6 m/s, in an orbit with a radius of 0.05 nm. (Although the electron and proton really orbit around their common center of mass, the center of mass is very close to the proton, since it is 2000 times more massive. For this problem, assume the proton is stationary.) In homework problem 15-a4, p. 170, you calculated the electric current created.
(a) Now estimate the magnetic field created at the center of the atom by the electron. We are treating the circling electron as a current loop, even though it's only a single particle. ✓
(b) Does the proton experience a nonzero force from the electron's magnetic field? Explain.

(c) Does the electron experience a magnetic field from the proton? Explain.
(d) Does the electron experience a magnetic field created by its own current? Explain.
(e) Is there an electric force acting between the proton and electron? If so, calculate it. √
(f) Is there a gravitational force acting between the proton and electron? If so, calculate it.
(g) An inward force is required to keep the electron in its orbit – otherwise it would obey Newton's first law and go straight, leaving the atom. Based on your answers to the previous parts, which force or forces (electric, magnetic and gravitational) contributes significantly to this inward force?
[Based on a problem by Arnold Arons.]

17-a15 The following data give the results of three experiments in which charged particles were released from the same point in space, and the forces on them were measured:

$$q_1 = 1\text{ C}\,, \quad \mathbf{v}_1 = 0\,, \quad \mathbf{F}_1 = (1\text{ N})\hat{\mathbf{y}}$$
$$q_2 = 1\text{ C}\,, \quad \mathbf{v}_2 = (1\text{ m/s})\hat{\mathbf{x}}\,, \quad \mathbf{F}_2 = (1\text{ N})\hat{\mathbf{y}}$$
$$q_3 = 1\text{ C}\,, \quad \mathbf{v}_3 = (1\text{ m/s})\hat{\mathbf{z}}\,, \quad \mathbf{F}_3 = 0$$

Determine the electric and magnetic fields. √

17-a16 If you put four times more current through a solenoid, how many times more energy is stored in its magnetic field? √

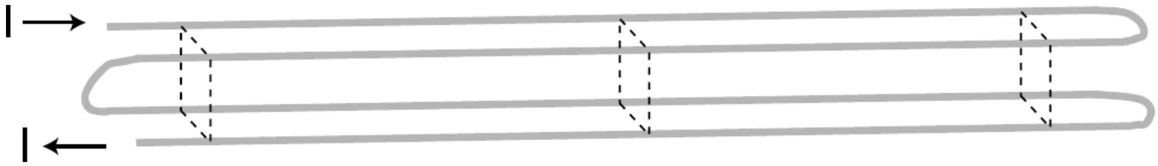
17-a17 A Helmholtz coil is defined as a pair of identical circular coils lying in parallel planes and separated by a distance, h, equal to their radius, b. (Each coil may have more than one turn of wire.) Current circulates in the same direction in each coil, so the fields tend to reinforce each other in the interior region. This configuration has the advantage of being fairly open, so that other apparatus can be easily placed inside and subjected to the field while remaining visible from the outside. The choice of $h = b$ results in the most uniform possible field near the center. A photograph of a Helmholtz coil is shown in example **??** on page **??**.
(a) Find the percentage drop in the field at the center of one coil, compared to the full strength at the center of the whole apparatus. √
(b) What value of h (not equal to b) would make this difference equal to zero? √

17-a18 The figure shows a nested pair of circular wire loops used to create magnetic fields. (The twisting of the leads is a practical trick for reducing the magnetic fields they contribute, so the fields are very nearly what we would expect for an ideal circular current loop.) The coordinate system below is to make it easier to discuss directions in space. One loop is in the $y-z$ plane, the other in the $x-y$ plane. Each of the loops has a radius of 1.0 cm, and carries 1.0 A in the direction indicated by the arrow.
(a) Calculate the magnetic field that would be produced by *one* such loop, at its center. √
(b) Describe the direction of the magnetic field that would be produced, at its center, by the loop in the $x-y$ plane alone.
(c) Do the same for the other loop.
(d) Calculate the magnitude of the magnetic field produced by the two loops in combination, at their common center. Describe its direction. √

17-a19 Four long wires are arranged, as shown, so that their cross-section forms a square, with connections at the ends so that current flows through all four before exiting. Note that the current is to the right in the two back wires, but to the left in the front wires. If the dimensions of the cross-sectional square (height and front-to-back) are b, find the magnetic field (magnitude and direction) along the long central axis. √

17-a20 In problem 17-a15, the three experiments gave enough information to determine both fields. Is it possible to design a procedure so that, using only two such experiments, we can always find **E** and **B**? If so, design it. If not, why not?

17-a21 Consider two solenoids, one of which is smaller so that it can be put inside the other. Assume they are long enough so that each one



Problem 17-a19.

only contributes significantly to the field inside itself, and the interior fields are nearly uniform. Consider the configuration where the small one is inside the big one with their currents circulating in the same direction, and a second configuration in which the currents circulate in opposite directions. Compare the energies of these configurations with the energy when the solenoids are far apart. Based on this reasoning, which configuration is stable, and in which configuration will the little solenoid tend to get twisted around or spit out?

17-a22 Consider two solenoids, one of which is smaller so that it can be put inside the other. Assume they are long enough to act like ideal solenoids, so that each one only contributes significantly to the field inside itself, and the interior fields are nearly uniform. Consider the configuration where the small one is partly inside and partly hanging out of the big one, with their currents circulating in the same direction. Their axes are constrained to coincide.
(a) Find the difference in the magnetic energy between the configuration where the solenoids are separate and the configuration where the small one is inserted into the big one. Your equation will include the length x of the part of the small solenoid that is inside the big one, as well as other relevant variables describing the two solenoids. $\checkmark$
(b) Based on your answer to part a, find the force acting

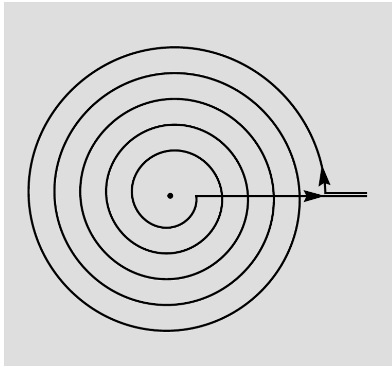
17-a23 Two long, parallel strips of thin metal foil form a configuration like a long, narrow sandwich. The air gap between them has height h, the width of each strip is w, and their length is ℓ. Each strip carries current I, and we assume for concreteness that the currents are in opposite directions, so that the magnetic force, F, between the strips is repulsive.
(a) Find the force in the limit of $w \gg h$. $\checkmark$
(b) Find the force in the limit of $w \ll h$, which is like two ordinary wires.
(c) Discuss the relationship between the two results.

17-a24 Suppose we are given a permanent magnet with a complicated, asymmetric shape. Describe how a series of measurements with a magnetic compass could be used to determine the strength and direction of its magnetic field at some point of interest. Assume that you are only able to see the direction to which the compass needle settles; you cannot measure the torque acting on it.

17-d1 Use the Biot-Savart law to show that the magnetic field of a long, straight wire is

$$|B| = \frac{2kI}{c^2R}$$

17-d2 Magnet coils are often wrapped in multiple layers. The figure shows the special case where the layers are all confined to a single plane, forming a spiral. Since the thickness of the wires (plus their insulation) is fixed, the spiral that results is a mathematical type known as an Archimedean spiral, in which the turns are evenly spaced. The equation of the spiral is $r = w\theta$, where w is a constant. For a spiral that starts from $r = a$ and ends at $r = b$, show that the field at the center is given by $(kI/c^2w)\ln b/a$.



Problem 17-d2.

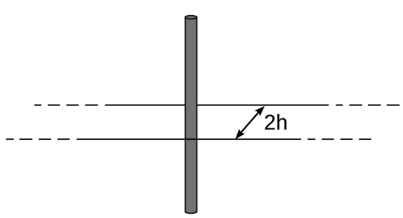
▷ Solution, p. 205

17-d3 Perform a calculation similar to the one in problem 17-d2, but for a logarithmic spiral, defined by $r = we^{u\theta}$, and show that the field is $B = (kI/c^2u)(1/a-1/b)$. Note that the solution to problem 17-d2 is given in the back of the book.

17-g1 A certain region of space has a magnetic field given by $\mathbf{B} = bx\hat{\mathbf{y}}$. Find the electric current flowing through the square defined by $z = 0$, $0 \le x \le a$, and $0 \le y \le a$.

$\checkmark$

17-g2 Verify Ampère's law in the case shown in the figure, assuming the known equation for the field of a wire. A wire carrying current I passes perpendicularly through the center of the rectangular Ampèrian surface. The length of the rectangle is infinite, so it's not necessary to compute the contributions of the ends.



Problem 17-g2.

18 Maxwell's equations and electromagnetic waves

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18.1 Maxwell's equations

The fundamental laws of physics governing electric and magnetic fields are Maxwell's equations, which state that for any closed surface, the fluxes through the surface are

$$\begin{aligned} \Phi_E &= 4\pi k q_{in} \qquad \text{and} \\ \Phi_B &= 0. \end{aligned}$$

For any surface that is not closed, the circulations around the edges of the surface are given by

$$\begin{aligned} \Gamma_E &= -\frac{\partial \Phi_B}{\partial t} \qquad \text{and} \\ c^2\Gamma_B &= \frac{\partial \Phi_E}{\partial t} + 4\pi k I_{through}. \end{aligned}$$

18.2 Electromagnetic waves

The most important result of Maxwell's equations is the existence of electromagnetic waves which propagate at the velocity of light — that's what light is. The waves are transverse, and the electric and magnetic fields are perpendicular to each other. There are no purely electric or purely magnetic waves; their amplitudes are always related to one another by $B = E/c$. They propagate in the right-handed direction given by the cross product $\mathbf{E} \times \mathbf{B}$, and carry momentum $p = U/c$.

18.3 Maxwell's equations in matter

A complete statement of Maxwell's equations in the presence of electric and magnetic materials is as follows:

$$\begin{aligned} \Phi_D &= q_{\text{free}} \\ \Phi_B &= 0 \\ \Gamma_E &= -\frac{\mathrm{d}\Phi_B}{\mathrm{d}t} \\ \Gamma_H &= \frac{\mathrm{d}\Phi_D}{\mathrm{d}t} + I_{\text{free}}, \end{aligned}$$

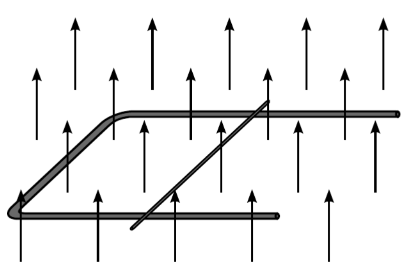
where the auxiliary fields $\mathbf{D}$ and $\mathbf{H}$ are defined as

$$\begin{aligned} \mathbf{D} &= \epsilon \mathbf{E} \qquad \text{and} \\ \mathbf{H} &= \frac{\mathbf{B}}{\mu}, \end{aligned}$$

and ϵ and μ are the permittivity and permeability of the substance.

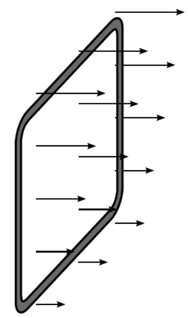
Problems

18-a1 A U-shaped wire makes electrical contact with a second, straight wire, which rolls along it to the right, as shown in the figure. The whole thing is immersed in a uniform magnetic field, which is perpendicular to the plane of the circuit. The resistance of the rolling wire is much greater than that of the U.
(a) Find the direction of the force on the wire based on conservation of energy.
(b) Verify the direction of the force using right-hand rules.
(c) Find magnitude of the force acting on the wire. There is more than one way to do this, but please do it using Faraday's law (which works even though it's the Ampèrian surface itself that is changing, rather than the field). ✓
(d) Consider how the answer to part a would have changed if the direction of the field had been reversed, and also do the case where the direction of the rolling wire's motion is reversed. Verify that this is in agreement with your answer to part c.



Problem 18-a1.

18-a2 A wire loop of resistance R and area A, lying in the $y-z$ plane, falls through a nonuniform magnetic field $\mathbf{B} = kz\hat{\mathbf{x}}$, where k is a constant. The z axis is vertical.
(a) Find the direction of the force on the wire based on conservation of energy.
(b) Verify the direction of the force using right-hand rules.
(c) Find the magnetic force on the wire.



Problem 18-a2.

✓

18-a3 (a) For each term appearing on the right side of Maxwell's equations, give an example of an everyday situation it describes.
(b) Most people doing calculations in the SI system of units don't use k and k/c^2. Instead, they express everything in terms of the constants

$$\epsilon_o = \frac{1}{4\pi k} \qquad \text{and}$$
$$\mu_o = \frac{4\pi k}{c^2}.$$

Rewrite Maxwell's equations in terms of these constants, eliminating k and c everywhere.

18-a4 The circular parallel-plate capacitor shown in the figure is being charged up over time, with the voltage difference across the plates varying as $V = st$, where s is a constant. The plates have radius b, and the distance between them is d. We assume $d \ll b$, so that the electric field between the plates is uniform, and parallel to the axis. Find the induced magnetic field at a point between the plates, at a distance R from the axis.

✓

18-d1 A charged particle is in motion at speed v, in a region of vacuum through which an electromagnetic wave is passing. In what direction should the particle be moving in order to minimize the total force acting on it? Consider both possibilities for the sign of the charge. (Based on a problem by David J. Raymond.)

18-d2 (a) Prove that in an electromagnetic plane wave, half the energy is in the electric field and half in the magnetic field.

(b) Based on your result from part a, find the proportionality constant in the relation $\mathrm{d}\mathbf{p} \propto \mathbf{E} \times \mathbf{B}\,\mathrm{d}v$, where $\mathrm{d}\mathbf{p}$ is the momentum of the part of a plane light wave contained in the volume $\mathrm{d}v$. The vector $\mathbf{E} \times \mathbf{B}$, multiplied by the appropriate constant, is known as the Poynting vector, and even outside the context of an electromagnetic plane wave, it can be interpreted as a momentum density or rate of energy flow. (To do this problem, you need to know the relativistic relationship between the energy and momentum of a beam of light from problem **??** on p. **??**.) √

18-d3 (a) A beam of light has cross-sectional area A and power P, i.e., P is the number of joules per second that enter a window through which the beam passes. Find the energy density U/v in terms of P, A, and universal constants.
(b) Find $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{B}}$, the amplitudes of the electric and magnetic fields, in terms of P, A, and universal constants (i.e., your answer should *not* include U or v). You will need the result of problem 18-d2a. A real beam of light usually consists of many short wavetrains, not one big sine wave, but don't worry about that. √
(c) A beam of sunlight has an intensity of $P/A = 1.35 \times 10^3$ W/m^2, assuming no clouds or atmospheric absorption. This is known as the solar constant. Compute $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{B}}$, and compare with the strengths of static fields you experience in everyday life: $E \sim 10^6$ V/m in a thunderstorm, and $B \sim 10^{-3}$ T for the Earth's magnetic field. √

18-d4 Electromagnetic waves are supposed to have their electric and magnetic fields perpendicular to each other. (Throughout this problem, assume we're talking about waves traveling through a vacuum, and that there is only a single sine wave traveling in a single direction, not a superposition of sine waves passing through each other.) Suppose someone claims they can make an electromagnetic wave in which the electric and magnetic fields lie in the same plane. Prove that this is impossible based on Maxwell's equations.

18-d5 A positively charged particle is released from rest at the origin at $t = 0$, in a region of vacuum through which an electromagnetic wave is passing. The particle accelerates in response to the wave. In this region of space, the wave varies as $\mathbf{E} = \hat{\mathbf{x}}\tilde{E}\sin\omega t$, $\mathbf{B} = \hat{\mathbf{y}}\tilde{B}\sin\omega t$, and we assume that the particle has a relatively large value of m/q, so that its response to the wave is sluggish, and it never ends up moving at any speed comparable to the speed of light. Therefore we don't have to worry about the spatial variation of the wave; we can just imagine that these are uniform fields imposed by some external mechanism on this region of space.
(a) Find the particle's coordinates as functions of time. √
(b) Show that the motion is confined to $-z_{max} \le z \le z_{max}$, where $z_{max} = 1.101\left(q^2\tilde{E}\tilde{B}/m^2\omega^3\right)$.

18-d6 If you watch a movie played backwards, some vectors reverse their direction. For instance, people walk backwards, with their velocity vectors flipped around. Other vectors, such as forces, keep the same direction, e.g., gravity still pulls down. An electric field is another example of a vector that doesn't turn around: positive charges are still positive in the time-reversed universe, so they still make diverging electric fields, and likewise for the converging fields around negative charges.
(a) How does the momentum of a material object behave under time-reversal?
(b) The laws of physics are still valid in the time-reversed universe. For example, show that if two material objects are interacting, and momentum is conserved, then momentum is still conserved in the time-reversed universe.
(c) Discuss how currents and magnetic fields would behave under time reversal.
(d) Similarly, show that the equation $\mathrm{d}\mathbf{p} \propto \mathbf{E}\times\mathbf{B}$ is still valid under time reversal.

▷ Solution, p. 205

18-d7 This problem is a more advanced exploration of the time-reversal ideas introduced in problem 18-d6.
(a) In that problem, we assumed that charge did not flip its sign under time reversal. Suppose we make the opposite assumption, that charge *does* change its sign. This is an idea introduced by Richard Feynman: that antimatter is really matter traveling backward in time! Determine the time-reversal properties of $\mathbf{E}$ and $\mathbf{B}$ under this new assumption, and show that $\mathrm{d}\mathbf{p} \propto \mathbf{E}\times\mathbf{B}$ is still valid under time-reversal.
(b) Show that Maxwell's equations are time-reversal symmetric, i.e., that if the fields $\mathbf{E}(x, y, z, t)$ and $\mathbf{B}(x, y, z, t)$ satisfy Maxwell's equations, then so do $\mathbf{E}(x, y, z, -t)$ and $\mathbf{B}(x, y, z, -t)$. Demonstrate this under both possible assumptions about charge, $q \rightarrow q$ and $q \rightarrow -q$.

18-g1 (a) Figure **??** on page **??** shows a hollow sphere with $\mu/\mu_o = x$, inner radius a, and outer radius b, which has been subjected to an external field $\mathbf{B}_o$. Finding the fields on the exterior, in the shell, and on the interior requires finding a set of fields that satisfies five boundary conditions: (1) far from the sphere, the field must approach the constant $\mathbf{B}_o$; (2) at the outer surface of the sphere, the field must have $\mathbf{H}_{\parallel,1} = \mathbf{H}_{\parallel,2}$, as discussed on page **??**; (3) the same constraint applies at the inner surface of the sphere; (4) and (5) there is an additional constraint on the fields at the inner and outer surfaces, as found in problem **??**. The goal of this problem is to find the solution for the fields, and from it, to prove that the interior field is uniform, and given by

$$\mathbf{B} = \left[\frac{9x}{(2x+1)(x+2) - 2\frac{a^3}{b^3}(x-1)^2}\right]\mathbf{B}_o.$$

This is a very difficult problem to solve from first principles, because it's not obvious what form the fields should have, and if you hadn't been told, you probably wouldn't have guessed that the interior field would be uniform. We could, however, guess that once the sphere becomes polarized by the external field, it would become a dipole, and at $r \gg b$, the field would be a uniform field superimposed on the field of a dipole. It turns out that even close to the sphere, the solution has exactly this form. In order to complete the solution, we need to find the field in the shell ($a < r < b$), but the only way this field could match up with the detailed angular variation of the interior and exterior fields would be if it was also a superposition of a uniform field with a dipole field. The final result is that we have four unknowns: the strength of the dipole component of the external field, the strength of the uniform and dipole components of the field within the shell, and the strength of the uniform interior field. These four unknowns are to be determined by imposing constraints (2) through (5) above.
(b) Show that the expression from part a has physically reasonable behavior in its dependence on x and a/b.

⋆⋆

19 LRC circuits

This is not a textbook. It's a book of problems meant to be used along with a textbook. Although each chapter of this book starts with a brief summary of the relevant physics, that summary is not meant to be enough to allow the reader to actually learn the subject from scratch. The purpose of the summary is to show what material is needed in order to do the problems, and to show what terminology and notation are being used.

19.1 Complex numbers

For a more detailed treatment of complex numbers, see ch. 3 of James Nearing's free book at physics.miami.edu/nearing/mathmethods/.

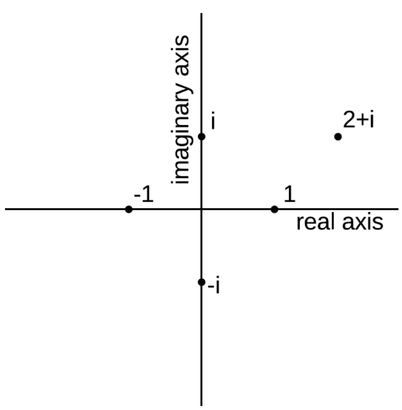


Figure 19.1: Visualizing complex numbers as points in a plane.

We assume there is a number, i, such that $i^2 = -1$. The square roots of -1 are then i and $-i$. (In electrical engineering work, where i stands for current, j is sometimes used instead.) This gives rise to a number system, called the complex numbers, containing the real numbers as a subset. Any complex number z can be written in the form $z = a+bi$, where a and b are real, and a and b are then referred to as the real and imaginary parts of z. A number with a zero real part is called an imaginary number. The complex numbers can be visualized as a plane, with the real number line placed horizontally like the x axis of the familiar $x-y$ plane, and the imaginary numbers running along the y axis. The complex numbers are complete in a way that the real numbers aren't: every nonzero complex number has two square roots. For example, 1 is a real number, so it is also a member of the complex numbers, and its square roots are -1 and 1. Likewise, -1 has square roots i and $-i$, and the number i has square roots $1/\sqrt{2}+i/\sqrt{2}$ and $-1/\sqrt{2}-i/\sqrt{2}$.

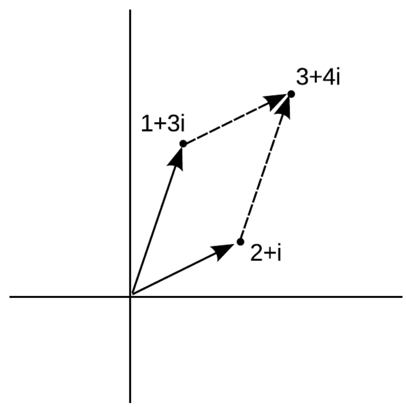


Figure 19.2: Addition of complex numbers is just like addition of vectors, although the real and imaginary axes don't actually represent directions in space.

Complex numbers can be added and subtracted by adding or subtracting their real and imaginary parts. Geometrically, this is the same as vector addition.

The complex numbers $a+bi$ and $a-bi$, lying at equal distances above and below the real axis, are called complex conjugates. The results of the quadratic formula are either both real, or complex conjugates of each other. The complex conjugate of a number z is notated as $\bar{z}$ or z^*.

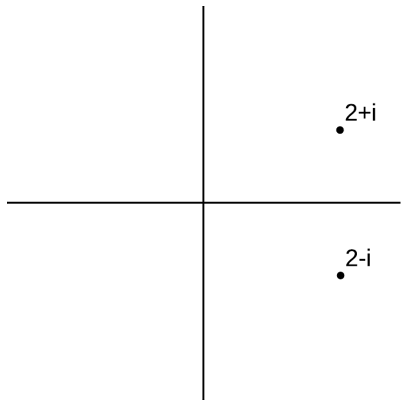


Figure 19.3: A complex number and its conjugate.

The complex numbers obey all the same rules of arithmetic as the reals, except that they can't be ordered along a single line. That is, it's not possible to say whether one complex number is greater than another. We can compare them in terms of their magnitudes (their distances from the origin), but two distinct complex numbers may have the same magnitude, so, for example, we can't say whether 1 is greater than i or i is greater than 1.

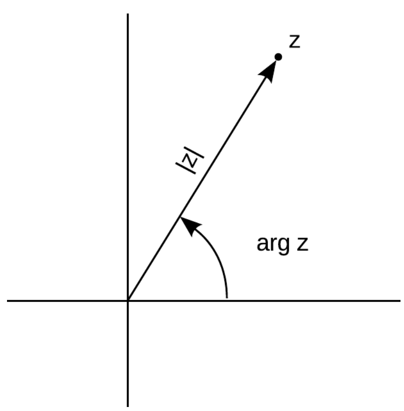


Figure 19.4: A complex number can be described in terms of its magnitude and argument.

There is a nice interpretation of complex multiplication. We define the argument of a complex number as its angle in the complex plane, measured counterclockwise from the positive real axis. Multiplying two complex numbers then corresponds to multiplying their magnitudes, and adding their arguments.

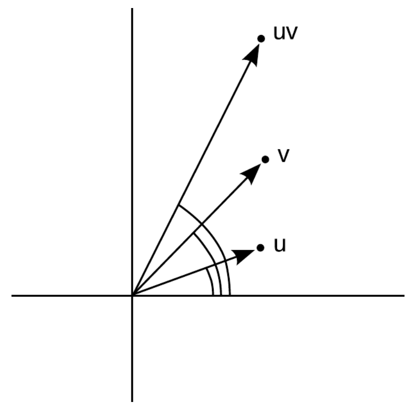


Figure 19.5: The argument of uv is the sum of the arguments of u and v.

Having expanded our horizons to include the complex numbers, it's natural to want to extend functions we knew and loved from the world of real numbers so that they can also operate on complex numbers. The only really natural way to do this in general is to use Taylor series. A particularly beautiful thing happens with the functions e^x, $\sin x$, and $\cos x$:

$$\begin{aligned}
e^x &= 1 + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \ldots \\
\cos x &= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \ldots \\
\sin x &= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \ldots
\end{aligned}$$

If $x = i\phi$ is an imaginary number, we have

$$e^{i\phi} = \cos\phi + i\sin\phi,$$

a result known as Euler's formula. The geometrical interpretation in the complex plane is shown in figure 19.6.

Although the result may seem like something out of a freak show at first, applying the definition of the exponential function makes it clear how natural it is:

$$e^x = \lim_{n\to\infty} \left(1+\frac{x}{n}\right)^n .$$

When $x = i\phi$ is imaginary, the quantity $(1 + i\phi/n)$ represents a number lying just above 1 in the complex plane. For large n, $(1 + i\phi/n)$ becomes very close to the unit circle, and its argument is the small angle ϕ/n. Raising this number to the nth power multiplies its argument by n, giving a number with an argument of ϕ.

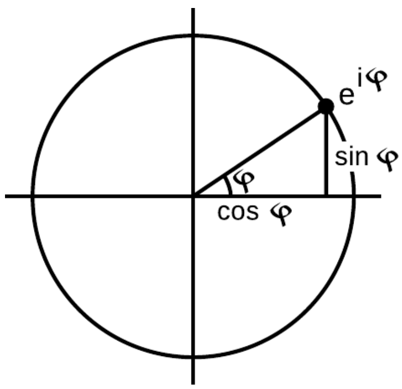


Figure 19.6: The complex number $e^{i\phi}$ lies on the unit circle.

Sinusoidal functions have a remarkable property, which is that if you add two different sinusoidal functions having the same frequency, the result is also a sinusoid with that frequency. For example, $\cos\omega t + \sin\omega t = \sqrt{2}\sin(\omega t + \pi/4)$, which can be proved using trig identities. The trig identities can get very cumbersome, however, and there is a much easier technique involving complex numbers.

Figure 19.7 shows a useful way to visualize what's going on. When a circuit is oscillating at a frequency ω, we use points in the plane to represent sinusoidal functions with various phases and amplitudes.

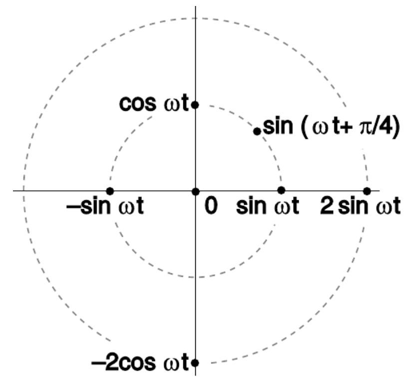


Figure 19.7: Representing functions with points in polar coordinates.

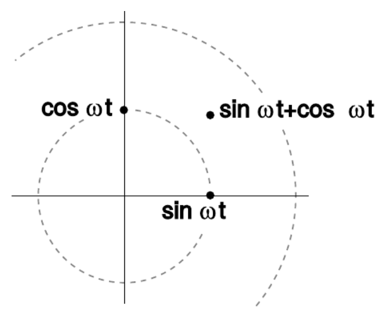


Figure 19.8: Adding two sinusoidal functions.

The simplest examples of how to visualize this in polar coordinates are ones like $\cos\omega t + \cos\omega t = 2\cos\omega t$, where everything has the same phase, so all the points lie along a single line in the polar plot, and addition is just like adding numbers on the number line. The less trivial example $\cos\omega t + \sin\omega t = \sqrt{2}\sin(\omega t + \pi/4)$, can be visualized as in figure 19.8.

Figure 19.8 suggests that all of this can be tied together nicely if we identify our plane with the plane of complex numbers. For example, the complex numbers 1 and i represent the functions $\sin\omega t$ and $\cos\omega t$.

19.2 LRC circuits

As discussed previously, capacitance, C, is defined as

$$U_C = \frac{1}{2C}q^2.$$

Similarly, inductance, L, is defined as

$$U_L = \frac{L}{2}I^2,$$

and measured in units of henries. The magnitude of the voltage difference across a capacitor or inductor is given by

$$V_C = \frac{q}{C}$$

or

$$V_L = L\frac{\mathrm{d}I}{\mathrm{d}t}.$$

In the equation for the inductor, the direction of the voltage drop (plus or minus sign) is such that the inductor resists the change in current.

A series LRC circuit exhibits *oscillation*, and, if driven by an external voltage, resonates. The Q of the circuit relates to the resistance value. For large Q, the resonant frequency is

$$\omega \approx \frac{1}{\sqrt{LC}}.$$

A series RC or RL circuit exhibits exponential *decay*,

$$q = q_\mathrm{o} \exp\left(-\frac{t}{RC}\right)$$

or

$$I = I_\mathrm{o} \exp\left(-\frac{R}{L}t\right),$$

and the quantity RC or L/R is known as the time constant.

When driven by a sinusoidal AC voltage with amplitude $\tilde{V}$, a capacitor, resistor, or inductor responds with a current having amplitude

$$\tilde{I} = \frac{\tilde{V}}{Z},$$

where the *impedance*, Z, is a frequency-dependent quantity having units of ohms. In a capacitor, the current has a phase that is 90° ahead of the voltage, while in an inductor the current is 90° behind. We can represent these phase relationships by defining the impedances as complex numbers:

$$Z_C = -\frac{i}{\omega C}$$
$$Z_R = R$$
$$Z_L = i\omega L$$

The arguments of the complex impedances are to be interpreted as phase relationships between the oscillating voltages and currents. The complex impedances defined in this way combine in series and parallel according to the same rules as resistances.

When a voltage source is driving a load through a transmission line, the maximum power is delivered to the load when the impedances of the line and the load are matched.

Problems

19-a1 Find $\arg i$, $\arg(-i)$, and $\arg 37$, where $\arg z$ denotes the argument of the complex number z.

▷ Solution, p. 205

19-a2 Visualize the following multiplications in the complex plane using the interpretation of multiplication in terms of multiplying magnitudes and adding arguments: $(i)(i) = -1$, $(i)(-i) = 1$, $(-i)(-i) = -1$.

▷ Solution, p. 205

19-a3 If we visualize z as a point in the complex plane, how should we visualize $-z$? What does this mean in terms of arguments? Give similar interpretations for z^2 and $\sqrt{z}$.

▷ Solution, p. 205

19-a4 Find four different complex numbers z such that $z^4 = 1$.

▷ Solution, p. 205

19-a5 Compute the following. For the final two, use the magnitude and argument, not the real and imaginary parts.

$$|1+i| \quad , \quad \arg(1+i) \quad , \quad \left|\frac{1}{1+i}\right| \quad , \quad \arg\left(\frac{1}{1+i}\right)$$

From these, find the real and imaginary parts of $1/(1+i)$.

▷ Solution, p. 205

19-d1 (a) Use complex number techniques to rewrite the function $f(t) = 4\sin\omega t + 3\cos\omega t$ in the form $A\sin(\omega t + \delta)$. ✓
(b) Verify the result using the trigonometric identity $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$.

19-d2 Calculate the quantity i^i (i.e., find its real and imaginary parts).

✓

19-g1 (a) Show that the equation $V_L = L\,\mathrm{d}I/\,\mathrm{d}t$ has the right units.
(b) Verify that RC has units of time.
(c) Verify that L/R has units of time.

19-g2 Find the inductance of two identical inductors in parallel.

19-g3 The wires themselves in a circuit can have resistance, inductance, and capacitance. Would "stray" inductance and capacitance be most important for low-frequency or for high-frequency circuits? For simplicity, assume that the wires act like they're in *series* with an inductor or capacitor.

19-g4 Starting from the relation $V = L\,\mathrm{d}I/\,\mathrm{d}t$ for the voltage difference across an inductor, show that an inductor has an impedance equal to $L\omega$.

19-g5 If an FM radio tuner consisting of an LRC circuit contains a 1.0 μH inductor, what range of capacitances should the variable capacitor be able to provide?

✓

19-g6 (a) Find the parallel impedance of a 37 kΩ resistor and a 1.0 nF capacitor at $f = 1.0\times10^4$ Hz. ✓
(b) A voltage with an amplitude of 1.0 mV drives this impedance at this frequency. What is the amplitude of the current drawn from the voltage source, what is the current's phase angle with respect to the voltage, and does it lead the voltage, or lag behind it?

✓

19-g7 A series LRC circuit consists of a 1.000 Ω resistor, a 1.000 F capacitor, and a 1.000 H inductor. (These are not particularly easy values to find on the shelf at Radio Shack!)
(a) Plot its impedance as a point in the complex plane for each of the following frequencies: ω=0.250, 0.500, 1.000, 2.000, and 4.000 Hz.
(b) What is the resonant angular frequency, ω_{res}, and how does this relate to your plot? ✓
(c) What is the resonant frequency f_{res} corresponding to your answer in part b?

✓

19-g8 At a frequency ω, a certain series LR circuit has an impedance of $1\ \Omega + (2\ \Omega)i$. Suppose that instead we want to achieve the same impedance using two circuit elements in parallel. What must the elements be?

19-g9 (a) In a series LC circuit driven by a DC voltage ($\omega = 0$), compare the energy stored in the inductor to the energy stored in the capacitor.
(b) Carry out the same comparison for an LC circuit that is oscillating freely (without any driving voltage).
(c) Now consider the general case of a series LC circuit driven by an oscillating voltage at an arbitrary frequency. Let $\overline{U_L}$ and be the average energy stored in the inductor, and similarly for $\overline{U_C}$. Define a quantity $u = \overline{U_C}/(\overline{U_L} + \overline{U_C})$, which can be interpreted as the capacitor's average share of the energy, while $1 - u$ is the inductor's average share. Find u in terms of L, C, and ω, and sketch a graph of u and $1 - u$ versus ω. What happens at resonance? Make sure your result is consistent with your answer to part a. $\checkmark$

Answers

1-a1

$$134\ \text{mg} \times \frac{10^{-3}\ \text{g}}{1\ \text{mg}} \times \frac{10^{-3}\ \text{kg}}{1\ \text{g}} = 1.34 \times 10^{-4}\ \text{kg}$$

1-d1 (a) Let's do 10.0 g and 1000 g. The arithmetic mean is 505 grams. It comes out to be 0.505 kg, which is consistent. (b) The geometric mean comes out to be 100 g or 0.1 kg, which is consistent. (c) If we multiply meters by meters, we get square meters. Multiplying grams by grams should give square grams! This sounds strange, but it makes sense. Taking the square root of square grams (g^2) gives grams again. (d) No. The superduper mean of two quantities with units of grams wouldn't even be something with units of grams! Related to this shortcoming is the fact that the superduper mean would fail the kind of consistency test carried out in the first two parts of the problem.

1-d2 (a) They're all defined in terms of the ratio of side of a triangle to another. For instance, the tangent is the length of the opposite side over the length of the adjacent side. Dividing meters by meters gives a unitless result, so the tangent, as well as the other trig functions, is unitless. (b) The tangent function gives a unitless result, so the units on the right-hand side had better cancel out. They do, because the top of the fraction has units of meters squared, and so does the bottom.

1-d3 The final line is supposed to be an equation for the height, so the units of the expression on the right-hand side had better equal meters. The pi and the 3 are unitless, so we can ignore them. In terms of units, the final becomes

$$\text{m} = \frac{\text{m}^2}{\text{m}^3} = \frac{1}{\text{m}}.$$

This is false, so there must be a mistake in the algebra. The units of lines 1, 2, and 3 check out, so the mistake must be in the step from line 3 to line 4. In fact the result should have been

$$h = \frac{3V}{\pi r^2}.$$

Now the units check: $\text{m} = \text{m}^3/\text{m}^2$.

1-j1 The proportionality $V \propto L^3$ can be restated in terms of ratios as $V_1/V_2 = (L_1/L_2)^3 = (1/10)^3 = 1/1000$, so scaling down the linear dimensions by a factor of 1/10 reduces the volume by 1/1000, to a milliliter.

1-j2

$$1\ \text{mm}^2 \times \left(\frac{1\ \text{cm}}{10\ \text{mm}}\right)^2 = 10^{-2}\ \text{cm}^2$$

1-j3 The bigger scope has a diameter that's ten times greater. Area scales as the square of the linear dimensions, so $A \propto d^2$, or in the language of ratios $A_1/A_2 = (d_1/d_2)^2 = 100$. Its light-gathering power is a hundred times greater.

1-j4 The cone of mixed gin and vermouth is the same shape as the cone of vermouth, but its linear dimensions are doubled. Translating the proportionality $V \propto L^3$ into an equation about ratios, we have $V_1/V_2 = (L_1/L_2)^3 = 8$. Since the ratio of the whole thing to the vermouth is 8, the ratio of gin to vermouth is 7.

1-k1 Since they differ by two steps on the Richter scale, the energy of the bigger quake is 10^4 times greater. The wave forms a hemisphere, and the surface area of the hemisphere over which the energy is spread is proportional to the square of its radius, $A \propto r^2$, or $r \propto \sqrt{A}$, which means $r_1/r_2 = \sqrt{A_1/A_2}$. If the amount of vibration was the same, then the surface areas must be in the ratio $A_1/A_2 = 10^4$, which means that the ratio of the radii is 10^2.

1-p1 Directly guessing the number of jelly beans would be like guessing volume directly. That would be a mistake. Instead, we start by estimating the linear dimensions, in units of beans. The contents of the jar look like they're about 10 beans deep. Although the jar is a cylinder, its exact geometrical shape doesn't really matter for the purposes of our order-of-magnitude estimate. Let's pretend it's a rectangular jar. The horizontal dimensions are also

something like 10 beans, so it looks like the jar has about $10 \times 10 \times 10$ or $\sim 10^3$ beans inside.

1-q1 Let's estimate the Great Wall's volume, and then figure out how many bricks that would represent. The wall is famous because it covers pretty much all of China's northern border, so let's say it's 1000 km long. From pictures, it looks like it's about 10 m high and 10 m wide, so the total volume would be $10^6\ \text{m} \times 10\ \text{m} \times 10\ \text{m} = 10^8\ \text{m}^3$. If a single brick has a volume of 1 liter, or $10^{-3}\ \text{m}^3$, then this represents about 10^{11} bricks. If one person can lay 10 bricks in an hour (taking into account all the preparation, etc.), then this would be 10^{10} man-hours.

2-k1 $\Delta x = \frac{1}{2}at^2$, so for a fixed value of Δx, we have $t \propto 1/\sqrt{a}$. Translating this into the language of ratios gives $t_M/t_E = \sqrt{a_E/a_M} = \sqrt{3} = 1.7$.

2-n4 (a) Other than w, the only thing with units that can occur in our answer is g. If we want to combine a distance and an acceleration to produce a time, the only way to do so is like $\sqrt{w/g}$, possibly multiplied by a unitless constant.

(b) It is convenient to introduce the notations L for the length of one side of the vee and h for the height, so that $L^2 = w^2 + h^2$. The acceleration is $a = g \sin\theta = gh/L$. To travel a distance L with this acceleration takes time

$$t = \sqrt{2L/a} = \sqrt{\left(\frac{2w}{g}\right)\left(\frac{h}{w} + \frac{w}{h}\right)}.$$

Let $x = h/w$. For a fixed value of w, this time is an increasing function of $x + 1/x$, so we want the value of x that minimizes this expression. Taking the derivative and setting it equal to zero gives $x = 1$, or $h = w$. In other words, the time is minimized if the angle is 45°.

(c) Plugging $x = 1$ back in, we have $t^* = 2t = 4\sqrt{w/g}$, so the unitless factor was 4.

2-p2 (a) Solving for $\Delta x = \frac{1}{2}at^2$ for a, we find $a = 2\Delta x/t^2 = 5.51\ \text{m/s}^2$. (b) $v = \sqrt{2a\Delta x} = 66.6$ m/s. (c) The actual car's final velocity is less than that of the idealized constant-acceleration car. If the real car and the idealized car covered the quarter mile in the same time but the real car was moving more slowly at the end than the idealized one, the real car must have been going faster than the idealized car at the beginning of the race. The real car apparently has a greater acceleration at the beginning, and less acceleration at the end. This make sense, because every car has some maximum speed, which is the speed beyond which it cannot accelerate.

3-j2 The boat's velocity relative to the land equals the vector sum of its velocity with respect to the water and the water's velocity with respect to the land,

$$\mathbf{v}_{BL} = \mathbf{v}_{BW} + \mathbf{v}_{WL}.$$

If the boat is to travel straight across the river, i.e., along the y axis, then we need to have $\mathbf{v}_{BL,x} = 0$. This x component equals the sum of the x components of the other two vectors,

$$\mathbf{v}_{BL,x} = \mathbf{v}_{BW,x} + \mathbf{v}_{WL,x},$$

or

$$0 = -|\mathbf{v}_{BW}| \sin\theta + |\mathbf{v}_{WL}|.$$

Solving for θ, we find

$$\sin\theta = |\mathbf{v}_{WL}|/|\mathbf{v}_{BW}|,$$

so

$$\theta = \sin^{-1} \frac{|\mathbf{v}_{WL}|}{|\mathbf{v}_{BW}|}.$$

4-a1 (a) The force of gravity on an object can't just be g, both because g doesn't have units of force and because the force of gravity is different for different objects.

(b) The force of gravity on an object can't just be m either. This again has the wrong units, and it also can't be right because it should depend on how strong gravity is in the region of space where the object is.

(c) If the object happened to be free-falling, then the only force acting on it would be gravity, so by Newton's second law, $a = F/m$, where F is the force that we're trying to find. Solving for F,

we have $F = ma$. But the acceleration of a free-falling object has magnitude g, so the magnitude of the force is mg. The force of gravity on an object doesn't depend on what else is happening to the object, so the force of gravity must also be equal to mg if the object doesn't happen to be free-falling.

4-a2 (a) This is a measure of the box's resistance to a change in its state of motion, so it measures the box's mass. The experiment would come out the same in lunar gravity.
(b) This is a measure of how much gravitational force it feels, so it's a measure of weight. In lunar gravity, the box would make a softer sound when it hit.
(c) As in part a, this is a measure of its resistance to a change in its state of motion: its mass. Gravity isn't involved at all.

4-a5 $a = \Delta v/\Delta t$, and also $a = F/m$, so

$$\begin{aligned}\Delta t &= \frac{\Delta v}{a}\\ &= \frac{m\Delta v}{F}\\ &= \frac{(1000\ \text{kg})(50\ \text{m/s} - 20\ \text{m/s})}{3000\ \text{N}}\\ &= 10\ \text{s}\end{aligned}$$

4-m1 (a) The swimmer's acceleration is caused by the water's force on the swimmer, and the swimmer makes a backward force on the water, which accelerates the water backward. (b) The club's normal force on the ball accelerates the ball, and the ball makes a backward normal force on the club, which decelerates the club. (c) The bowstring's normal force accelerates the arrow, and the arrow also makes a backward normal force on the string. This force on the string causes the string to accelerate less rapidly than it would if the bow's force was the only one acting on it. (d) The tracks' backward frictional force slows the locomotive down. The locomotive's forward frictional force causes the whole planet earth to accelerate by a tiny amount, which is too small to measure because the earth's mass is so great.

5-d10 (a) There is no theoretical limit on how much normal force F_N the climber can make on the walls with each foot, so the frictional force can be made arbitrarily large. This means that with any $\mu > 0$, we can always get the vertical forces to cancel. The theoretical minimum value of μ will be determined by the need for the horizontal forces to cancel, so that the climber doesn't pop out of the corner like a watermelon seed squeezed between two fingertips. The horizontal component of the frictional force is always less than the magnitude of the frictional force, which is turn is less than μF_N. To find the minimum value of μ, we set the static frictional force equal to μF_N.

Let the x axis be along the plane that bisects the two walls, let y be the horizontal direction perpendicular to x, and let z be vertical. Then cancellation of the forces in the z direction is not the limiting factor, for the reasons described above, and cancellation in y is guaranteed by symmetry, so the only issue is the cancellation of the x forces. We have $2F_s \cos(\theta/2) - 2F_N \sin(\theta/2) = 0$. Combining this with $F_s = \mu F_N$ results in $\mu = \tan(\theta/2)$.

(b) For $\theta = 0$, μ is very close to zero. That is, we can always theoretically stay stuck between two parallel walls, simply by pressing hard enough, even if the walls are made of ice or polished marble with a coating of WD-40. As θ gets close to 180°, μ blows up to infinity. We need at least some dihedral angle to do this technique, because otherwise we're facing a flat wall, and there is nothing to cancel the wall's normal force on our feet.

(c) The result is 99.0°, i.e., just a little wider than a right angle.

5-m1 (a) By Newton's third law, the forces are F and $-F$. Pick a coordinate system in which skater 1 moves in the negative x direction due to a force $-F$. Since the forces are constant, the accelerations are also constant, and the distances moved by their centers of mass are $\Delta x_1 = (1/2)a_1T^2$ and $\Delta x_2 = (1/2)a_2T^2$. The accelerations are $a_1 = -F/m_1$ and $a_2 = F/m_2$.

We then have

$$\begin{aligned}\ell_f - \ell_0 &= \Delta x_2 - \Delta x_1 \\ &= \frac{1}{2}F\left(\frac{1}{m_1}+\frac{1}{m_2}\right)T^2,\end{aligned}$$

resulting in

$$T = \sqrt{\frac{2(\ell_f-\ell_0)}{F\left(\frac{1}{m_1}+\frac{1}{m_2}\right)}}$$

(b)

$$\sqrt{\frac{\mathrm{m}}{\mathrm{N/kg}}} = \sqrt{\frac{\mathrm{m}}{\mathrm{kg}\cdot\mathrm{ms}^{-2}\mathrm{kg}^{-1}}} = \mathrm{s}$$

(c) If the force is bigger, we expect physically that they will reach arm's length more quickly. Mathematically, a bigger F on the bottom results in a smaller T.
(d) If one of the masses is very small, then $1/m_1+1/m_2$ gets very big, and T gets very small. This makes sense physically. If you flick a flea off of yourself, contact is broken very quickly.

7-m1

$$\begin{aligned}E_{total,i} &= E_{total,f} \\ PE_i + \text{heat}_i &= PE_f + KE_f + \text{heat}_f \\ \frac{1}{2}mv^2 &= PE_i - PE_f + \text{heat}_i - \text{heat}_f \\ &= -\Delta PE - \Delta\text{heat} \\ v &= \sqrt{2\left(\frac{-\Delta PE - \Delta\text{heat}}{m}\right)} \\ &= 6.4\ \text{m/s}\end{aligned}$$

8-a1 Momentum is a vector. The total momentum of the molecules is always zero, since the momenta in different directions cancal out on the average. Cooling changes individual molecular momenta, but not the total.

8-a2 By conservation of momentum, the total momenta of the pieces after the explosion is the same as the momentum of the firework before the explosion. However, there is no law of conservation of kinetic energy, only a law of conservation of energy. The chemical energy in the gunpowder is converted into heat and kinetic energy when it explodes. All we can say about the kinetic energy of the pieces is that their total is greater than the kinetic energy before the explosion.

8-g3 A conservation law is about addition: it says that when you add up a certain thing, the total always stays the same. Funkosity would violate the additive nature of conservation laws, because a two-kilogram mass would have twice as much funkosity as a pair of one-kilogram masses moving at the same speed.

8-m3 Let m be the mass of the little puck and $M = 2.3m$ be the mass of the big one. All we need to do is find the direction of the total momentum vector before the collision, because the total momentum vector is the same after the collision. Given the two components of the momentum vector $p_x = Mv$ and $p_y = mv$, the direction of the vector is $\tan^{-1}(p_y/p_x) = 23°$ counterclockwise from the big puck's original direction of motion.

9-d13 The moment of inertia is $I = \int r^2\,\mathrm{d}m$. Let the ring have total mass M and radius b. The proportionality

$$\frac{M}{2\pi} = \frac{\mathrm{d}m}{\mathrm{d}\theta}$$

gives a change of variable that results in

$$I = \frac{M}{2\pi}\int_0^{2\pi} r^2\,\mathrm{d}\theta.$$

If we measure θ from the axis of rotation, then $r = b\sin\theta$, so this becomes

$$I = \frac{Mb^2}{2\pi}\int_0^{2\pi}\sin^2\theta\,\mathrm{d}\theta.$$

The integrand averages to $1/2$ over the 2π range of integration, so the integral equals π. We therefore have $I = \frac{1}{2}Mb^2$. This is, as claimed, half the value for rotation about the symmetry axis.

9-g1 The pliers are not moving, so their angular momentum remains constant at zero, and the total torque on them must be zero. Not only that, but each half of the pliers must have zero total torque on it. This tells us that the magnitude of the torque at one end must be the same

as that at the other end. The distance from the axis to the nut is about 2.5 cm, and the distance from the axis to the centers of the palm and fingers are about 8 cm. The angles are close enough to 90° that we can pretend they're 90 degrees, considering the rough nature of the other assumptions and measurements. The result is (300 N)(2.5 cm) = (F)(8 cm), or $F = 90$ N.

10-d5 (a) If the expression $1 + by$ is to make sense, then by has to be unitless, so b has units of m^{-1}. The input to the exponential function also has to be unitless, so k also has of m^{-1}. The only factor with units on the right-hand side is P_o, so P_o must have units of pressure, or Pa.
(b)

$$\begin{aligned}\mathrm{d}P &= \rho g\,\mathrm{d}y \\ \rho &= \frac{1}{g}\frac{\mathrm{d}P}{\mathrm{d}y} \\ &= \frac{P_o}{g}e^{-ky}(-k - kby + b)\end{aligned}$$

(c) The three terms inside the parentheses on the right all have units of m^{-1}, so it makes sense to add them, and the factor in parentheses has those units. The units of the result from b then look like

$$\begin{aligned}\frac{\text{kg}}{\text{m}^3} &= \frac{\text{Pa}}{\text{m/s}^2}\text{m}^{-1} \\ &= \frac{\text{N/m}^2}{\text{m}^2/\text{s}^2} \\ &= \frac{\text{kg}\cdot\text{m}^{-1}\cdot\text{s}^{-2}}{\text{m}^2/\text{s}^2},\end{aligned}$$

which checks out.

11-a1 Newton's law of gravity depends on the inverse square of the distance, so if the two planets' masses had been equal, then the factor of $0.83/0.059 = 14$ in distance would have caused the force on planet c to be $14^2 = 2.0 \times 10^2$ times weaker. However, planet c's mass is 3.0 times greater, so the force on it is only smaller by a factor of $2.0 \times 10^2/3.0 = 65$.

11-d1 Newton's law of gravity is $F = GMm/r^2$. Both G and the astronaut's mass m are the same in the two situations, so $F \propto Mr^{-2}$. In terms of ratios, this is

$$\frac{F_c}{F_e} = \frac{M_c}{M_e}\left(\frac{r_c}{r_e}\right)^{-2}.$$

The result is 11 N.

11-d2 (a) The asteroid's mass depends on the cube of its radius, and for a given mass the surface gravity depends on r^{-2}. The result is that surface gravity is directly proportional to radius. Half the gravity means half the radius, or one eighth the mass. (b) To agree with a, Earth's mass would have to be 1/8 Jupiter's. We assumed spherical shapes and equal density. Both planets are at least roughly spherical, so the only way out of the contradiction is if Jupiter's density is significantly less than Earth's.

11-g1 Any fractional change in r results in double that amount of fractional change in $1/r^2$. For example, raising r by 1% causes $1/r^2$ to go down by very nearly 2%. A 27-day orbit is 1/13.5 of a year, so the fractional change in $1/r^2$ is

$$2 \times \frac{(4/13.5)\ \text{cm}}{3.84 \times 10^5\ \text{km}} \times \frac{1\ \text{km}}{10^5\ \text{cm}} = 1.5 \times 10^{-11}$$

11-j5 Newton's second law gives

$$F = m_D a_D,$$

where F is Ida's force on Dactyl. Using Newton's universal law of gravity, F= $Gm_I m_D/r^2$,and the equation $a = v^2/r$ for circular motion, we find

$$Gm_I m_D/r^2 = m_D v^2/r.$$

Dactyl's mass cancels out, giving

$$Gm_I/r^2 = v^2/r.$$

Dactyl's velocity equals the circumference of its orbit divided by the time for one orbit: $v = 2\pi r/T$. Inserting this in the above equation and solving for m_I, we find

$$m_I = \frac{4\pi^2 r^3}{GT^2},$$

so Ida's density is

$$\begin{aligned}\rho &= m_I/V \\ &= \frac{4\pi^2 r^3}{GVT^2}.\end{aligned}$$

11-m6 (a) Based on units, we must have $g = kG\lambda/y$, where k is a unitless universal constant.
(b) For the actual calculation, we have

$$\begin{aligned}g &= \int \mathrm{d}g_y \\ &= G\int \frac{\mathrm{d}m}{r^2}\cos\theta,\end{aligned}$$

where θ is the angle between the perpendicular and the $\mathbf{r}$ vector. Then $\mathrm{d}m = \lambda\,\mathrm{d}x$, $\cos\theta = y/r$, and $r = \sqrt{x^2+y^2}$, so

$$\begin{aligned}g &= G\int_{-\infty}^{\infty} \frac{\lambda\,\mathrm{d}x}{x^2+y^2}\cdot\frac{b}{\sqrt{x^2+y^2}} \\ &= G\lambda y\int_{-\infty}^{\infty}(x^2+y^2)^{-3/2}\,\mathrm{d}x.\end{aligned}$$

Even though this has limits of integration, this is an indefinite integral because it contains the variable y. It's nicer to clean this up by doing a change of variable to the unitless quantity $u = x/y$, giving

$$g = \frac{G\lambda}{y}\int_{-\infty}^{\infty}(u^2+1)^{-3/2}\,\mathrm{d}u.$$

The definite integral is the sort of thing that sane people these days will do using computer software. It equals 2. The result for the field is

$$g = \frac{2G\lambda}{y}.$$

14-a4 (a) Conservation of energy gives

$$\begin{aligned}U_A &= U_B + K_B \\ K_B &= U_A - U_B \\ \frac{1}{2}mv^2 &= e\Delta V \\ v &= \sqrt{\frac{2e\Delta V}{m}}\end{aligned}$$

(b) Plugging in numbers, we get 5.9×10^7 m/s. This is about 20% of the speed of light, so the nonrelativistic assumption was good to at least a rough approximation.

14-d1 By symmetry, the field is always directly toward or away from the center. We can therefore calculate it along the x axis, where $r = x$, and the result will be valid for any location at that distance from the center.

$$\begin{aligned}E &= -\frac{\mathrm{d}V}{\mathrm{d}x} \\ &= -\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{-1}e^{-x}\right) \\ &= x^{-2}e^{-x} + x^{-1}e^{-x}\end{aligned}$$

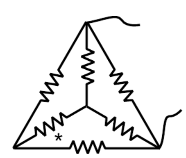
At small x, near the proton, the first term dominates, and the exponential is essentially 1, so we have $E \propto x^{-2}$, as we expect from the Coulomb force law. At large x, the second term dominates, and the field approaches zero faster than an exponential.

15-a1 $\Delta t = \Delta q/I = e/I = 0.16\ \mu$s

15-d5 In series, they give 11 kΩ. In parallel, they give $(1/1\text{ k}\Omega + 1/10\text{ k}\Omega)^{-1} = 0.9$ kΩ.

15-g2 It's much more practical to measure voltage differences. To measure a current, you have to break the circuit somewhere and insert the meter there, but it's not possible to disconnect the circuits sealed inside the board.

15-g11 The actual shape is irrelevant; all we care about is what's connected to what. Therefore, we can draw the circuit flattened into a plane. Every vertex of the tetrahedron is adjacent to every other vertex, so any two vertices to which we connect will give the same resistance. Picking two arbitrarily, we have this:



This is unfortunately a circuit that cannot be converted into parallel and series parts, and

that's what makes this a hard problem! However, we can recognize that by symmetry, there is zero current in the resistor marked with an asterisk. Eliminating this one, we recognize the whole arrangement as a triple parallel circuit consisting of resistances R, $2R$, and $2R$. The resulting resistance is $R/2$.

17-d2 Note that in the Biot-Savart law, the variable $\mathbf{r}$ is defined as a vector that points from the current to the point at which the field is being calculated, whereas in the polar coordinates used to express the equation of the spiral, the vector more naturally points the opposite way. This requires some fiddling with signs, which I'll suppress, and simply identify $\mathrm{d}\boldsymbol{\ell}$ with $\mathrm{d}\mathbf{r}$.

$$\mathbf{B} = \frac{kI}{c^2}\int \frac{\mathrm{d}\boldsymbol{\ell}\times\mathbf{r}}{r^3}$$

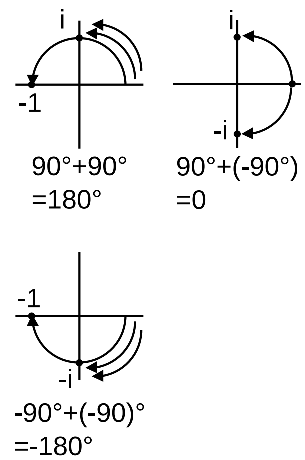
The vector $\mathrm{d}\mathbf{r}$ has components $\mathrm{d}x = w(\cos\theta - \theta\sin\theta)$ and $\mathrm{d}y = w(\sin\theta + \theta\cos\theta)$. Evaluating the vector cross product, and substituting θ/w for r, we find

$$\begin{aligned}
\mathbf{B} &= \frac{kI}{c^2 w}\int \frac{\theta(\cos\theta\sin\theta - \theta\sin^2\theta - \cos\theta\sin\theta - \theta\cos^2\theta)\,\mathrm{d}\theta}{\theta^3} \\
&= \frac{kI}{c^2 w}\int \frac{\mathrm{d}\theta}{\theta} \\
&= \frac{kI}{c^2 w}\ln\frac{\theta_2}{\theta_1} \\
&= \frac{kI}{c^2 w}\ln\frac{b}{a}
\end{aligned}$$

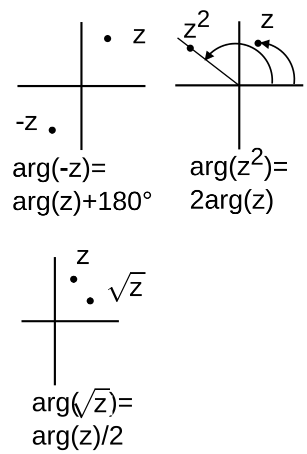
18-d6 (a) For a material object, $\mathbf{p} = m\mathbf{v}$. The velocity vector reverses itself, but mass is still positive, so the momentum vector is reversed.
(b) In the forward-time universe, conservation of momentum is $\mathbf{p}_{1,i} + \mathbf{p}_{2,i} = \mathbf{p}_{1,f} + \mathbf{p}_{2,f}$. In the backward-time universe, all the momenta are reversed, but that just negates both sides of the equation, so momentum is still conserved.

19-a1 $\arg i = 90°$, $\arg(-i) = -90°$, $\arg 37 = 0$

19-a2



19-a3



19-a4 1, i, -1, $-i$

19-a5

$$\begin{aligned}
&|1+i| = \sqrt{1^2+1^2} = \sqrt{2} \\
&\arg(1+i) = 45° \\
&\left|\frac{1}{1+i}\right| = \frac{1}{|1+i|} = 1/\sqrt{2} \\
&\arg\left(\frac{1}{1+i}\right) = -\arg(1+i) = -45°
\end{aligned}$$

The real part is $(1/\sqrt{2})\cos(-45°) = 1/2$. The imaginary part is $(1/\sqrt{2})\sin(-45°) = -1/2$.

Photo credits

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